# Chapter 22

# Nonlinear Dynamics of Plasmas

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## 22.1 Overview

In Chap. 20 we met our first example of a velocity space instability, the two stream instability, which illustrated the general principle that departures from Maxwellian equilibrium in velocity space in a collisionless plasma might be unstable and lead to the exponential growth of small amplitude waves, just as we found can happen for departures from spatial uniformity in a fluid. In Chap. 21, where we analyzed warm plasmas, we derived the dispersion relation for electrostatic waves in an unmagnetised plasma, we showed how Landau damping can damp the waves when the phase space density of the resonant particles diminishes with increasing speed, and we showed that in the opposite case of an increasing phase space density the waves can grow at the expense of the energies of near-resonant particles (provided the Penrose criterion is satisfied). In this chapter, we shall explore the back-reaction of the waves on the near-resonant particles. This back-reaction is a (weakly) nonlinear process, so we shall have to extend our analysis of the wave-particle interactions to include the leading nonlinearity.

This extension is called *quasilinear theory* or *weak turbulence theory*, and it allows us to follow the time development of the waves and the near-resonant particles simultaneously. We develop this formalism in Sec. 22.2 and verify that it enforces the laws of particle conservation, energy conservation, and momentum conservation. Our original development of the formalism is entirely in classical language and meshes nicely with the theory of electrostatic waves as presented in Chap. 21. In Sec. 22.3, we reformulate the theory in terms of the emission, absorption and scattering of wave quanta. Although waves in plasmas almost always entail large quantum occupation numbers and thus are highly classical, this quantum formulation of the classical theory has great computational and heuristic power (and as one would expect, despite the presence of Planck's constant $\hbar$ in the formalism, $\hbar$ nowhere appears in the final answers to problems). Our initial derivation and development of the formalism is restricted to the interaction of electrons with electrostatic waves, but we also describe how the formalism can be generalized to describe a multitude of wave modes and particle species interacting with each other. We also describe circumstances in which this formalism can fail, and the resonant particles can couple strongly, *not* to a broad-band distribution of incoherent waves (as the formalism presumes) but instead to one or a few individual, coherent modes. In Sec. 22.6 we explore an example.

In Sec. 22.4 we turn to our first illustrative application of quasilinear theory: to a warm electron beam propagating through a stable plasma. We show how the particle distribution function evolves so as to shut down the growth of the waves and we illustrate this by describing the problem of the isotropization of Galactic cosmic rays. Next, in Sec. 22.5, we consider parametric instabilities which are very important in the absorption of laser light in experimental studies of the compression of small deuterium-tritium pellets - a possible forerunner of a commercial nuclear fusion reactor. Finally, in Sec. 22.6 we return to ion acoustic solitons and explain how the introduction of dissipation can create a collisionless shock, similar to that found where the earth's bow shock meets the solar wind.

## 22.2 Quasilinear Theory in Classical Language

### 22.2.1 Classical Derivation of the Theory

In Chap. 21 we discovered that a distribution of hot electrons or ions can Landau damp a wave mode. We also showed that some distributions lead to exponential growth of the waves in time. Either way there is energy transfer between the waves and the particles. We now turn to the back-reaction of the waves on the near-resonant particles that damp or amplify them. For simplicity, we shall derive the back-reaction equations ("quasilinear theory") in the special case of electrons interacting with electrostatic Langmuir waves, and then shall assert the (rather obvious) generalization to protons or other ions and to interaction with other types of wave modes.

We begin with the electrons' one-dimensional distribution function $F_e(v, z, t)$ [Eq. (21.14)]. As in Chap. 21, we split $F_e$ into two parts, but we must do so more carefully here than there. The foundation for our split is a two-lengthscale expansion of the same sort as we used in developing geometric optics (Sec. 6.3): We introduce two disparate lengthscales, the short one being the typical reduced wavelength of a Langmuir wave $\lambdabar \sim 1/k$, and the long one being a scale $L \gg \lambdabar$ over which we perform spatial averages. Later, when applying our formalism to an inhomogeneous plasma, $L$ must be somewhat shorter than the spatial inhomogeneity scale but still $\gg \lambdabar$. In our present situation of a homogeneous background plasma, there can still be large-scale spatial inhomogeneities caused by the growth or damping of the wave modes by their interaction with the electrons, and we must choose $L$ somewhat smaller than the growth or damping length but still large compared to $\lambdabar$.

Our split of $F_e$ is into the spatial average of $F_e$ over the length $L$ (denoted $F_0$) plus a rapidly varying part that averages to zero (denoted $F_1$):

$$F_0 \equiv \langle F_e \rangle \ , \quad F_1 \equiv F_e - F_0 \ , \quad F_e = F_0 + F_1 \ . \tag{22.1}$$

(For simplicity, we omit the subscript $e$ from $F_0$ and $F_1$.)

The time evolution of $F_e$, and thence of $F_0$ and $F_1$, is governed by the one-dimensional Vlasov equation, in which we assume a uniform neutralizing ion background, no background magnetic field, and interaction with electrostatic waves. We cannot use the linearized Vlasov equation (21.15), which formed the foundation for all of Chap. 21, because the processes we wish to study are nonlinear. Rather, we must use the fully nonlinear Vlasov equation [Eq. (21.5) integrated over the irrelevant components $v_x$ and $v_y$ of velocity as in Eq. (21.14)]:

$$\frac{\partial F_e}{\partial t} + v\frac{\partial F_e}{\partial z} - \frac{eE}{m_e}\frac{\partial F_e}{\partial v} = 0 \ . \tag{22.2}$$

Here $E$ is the rapidly varying electric field associated with the waves.

Inserting $F_e = F_0 + F_1$ into this Vlasov equation, we obtain

$$\frac{\partial F_0}{\partial t} + v\frac{\partial F_0}{\partial z} - \frac{e}{m_e}\frac{\partial F_1}{\partial v}E + \frac{\partial F_1}{\partial t} + v\frac{\partial F_1}{\partial z} - \frac{e}{m_e}\frac{\partial F_0}{\partial v}E = 0 \ . \tag{22.3}$$

We then split this equation into two parts, its average over the large lengthscale $L$ and its remaining time-varying part.

The averaged part gets contributions only from the first three terms (since the last three are linear in $F_1$ and $E$, which have vanishing averages):

$$\frac{\partial F_0}{\partial t} + v\frac{\partial F_0}{\partial z} - \frac{e}{m_e}\left\langle\frac{\partial F_1}{\partial v}E\right\rangle = 0 \ . \tag{22.4}$$

This is an evolution equation for the averaged distribution $F_0$; the third, nonlinear term drives the evolution. This driving term is the only nonlinearity that that we shall keep in the quasilinear Vlasov equation.

The rapidly varying part of the Vlasov equation (22.3) is just the last three terms

$$\frac{\partial F_1}{\partial t} + v\frac{\partial F_1}{\partial z} - \frac{e}{m_e}\frac{\partial F_0}{\partial v}E = 0 \ , \tag{22.5}$$

plus a nonlinear term

$$-\frac{e}{m_e}\left(\frac{\partial F_1}{\partial v}E - \left\langle\frac{\partial F_1}{\partial v}E\right\rangle\right) \tag{22.6}$$

which we discard as being far smaller than the linear ones. If we were to keep this term, we would find that it can produce a "three-wave mixing", in which two electrostatic waves with different wave numbers $k_1$ and $k_2$ interact weakly to try to generate a third electrostatic wave with wave number $k_3 = k_1 \pm k_2$. We shall discuss such three-wave mixing in Sec. 22.3.3 below; for the moment we shall ignore it, and correspondingly shall discard the nonlinearity (22.6).

Equation (22.5) is the same linear evolution equation for $F_1$ as we developed and studied in Chap. 21. Here as there we bring its physics to the fore by decomposing into plane-wave, monochromatic modes; but here, by contrast with there, we shall be dealing with many modes and sums over the effects of many modes, so we must do the decomposition a little more carefully. The foundation for the decomposition is a spatial Fourier transform inside our averaging "box" of length $L$,

$$\tilde{F}_1(v,k,t) = \int_0^L e^{-ikz} F_1(v,z,t)dz \ , \quad \tilde{E}(k,t) = \int_0^L e^{-ikz} E(z,t)dz. \tag{22.7}$$

We take $F_1$ and $E$ to represent the physical quantities and thus to be real; this implies that $\tilde{F}_1(-k) = \tilde{F}_1^*(k)$ and similarly for $\tilde{E}$, so the inverse Fourier transforms are

$$F_1(v,z,t) = \int_{-\infty}^{\infty} e^{ikz} \tilde{F}_1(v,k,t)\frac{dk}{2\pi} \ , \quad E(z,t) = \int_{-\infty}^{\infty} e^{ikz} \tilde{E}(v,k,t)\frac{dk}{2\pi} \quad \text{for } 0 < z < L \ . \tag{22.8}$$

(This choice of how to do the mathematics corresponds to idealizing $F_1$ and $E$ as vanishing outside the box; alternatively we could treat them as though they were periodic with period $L$ and replace Eq. (22.8) by a sum over discrete values of $k$—multiples of $2\pi/L$.)

From our study of linearized waves in Chap. 20, we know that a mode with wave number $k$ will oscillate in time with some frequency $\omega(k)$ so

$$\tilde{F}_1 \propto e^{-i\omega(k)t} \quad \text{and} \quad \tilde{E} \propto e^{-i\omega(k)t} \ . \tag{22.9}$$

For simplicity, we assume that the mode propagates in the $+z$ direction; when studying modes traveling in the opposite direction, we just turn our $z$ axis around. In Sec. 22.2.4 we will generalize to three-dimensional situations and include all directions of propagation simultaneously. For simplicity, we also assume that for each wave number $k$ there is at most one mode type present (i.e, only a Langmuir wave or only an ion acoustic wave). With these simplifications, $\omega(k)$ is a unique function with $\omega_r > 0$ when $k > 0$. Notice that the reality of $E(z,t)$ implies [from the second of Eqs. (22.7)] $\tilde{E}(-k,t) = \tilde{E}^*(k,t)$ for all $t$, i.e. [cf. Eq. (22.9)] $\tilde{E}(-k,0)e^{-i\omega(-k)t} = \tilde{E}^*(k,0)e^{+i\omega^*(k)t}$ for all $t$, which in turn implies

$$\omega(-k) = -\omega^*(k) \ ; \quad \text{i.e.} \quad \omega_r(-k) = -\omega_r(k) \ , \quad \omega_i(-k) = \omega_i(k) \ . \tag{22.10}$$

This should be obvious physically: it says that, for our chosen conventions, both the negative $k$ and positive $k$ contributions to Eq. (22.8) propagate in the $+z$ direction, and both grow or are damped in time at the same rate. In general, $\omega(k)$ is determined by the Landau-contour dispersion relation (21.30). However, throughout Secs. 22.2–22.4 we shall specialize to weakly damped or growing Langmuir waves with phase velocities $\omega_r/k$ large compared to the rms electron speed:

$$v_{\rm ph} = \frac{\omega_r}{k} \gg v_{\rm rms} = \sqrt{\int v^2 F_0(v) dv} \ . \tag{22.11}$$

For such waves, from Eqs. (21.34), (21.35), and the first two lines of (21.37), we deduce the following explicit forms for the real and imaginary parts of $\omega$:

$$\omega_r = \omega_p \left(1 + \frac{3}{2}\frac{v_{\rm rms}^2}{(\omega_p/k)^2}\right) \quad \text{for } k > 0 \ , \tag{22.12}$$

$$\omega_i = \frac{\pi e^2}{2\epsilon_0 m_e}\frac{\omega_r}{k^2} F_0'(\omega_r/k) \quad \text{for } k > 0. \tag{22.13}$$

The linearized Vlasov equation (22.5) implies that the modes' amplitudes $\tilde{F}_1(v,k,t)$ and $\tilde{E}(k,t)$ are related by

$$\tilde{F}_1 = \frac{ie}{m_e}\frac{\partial F_0/\partial v}{(\omega - kv)}\tilde{E} \; . \tag{22.14}$$

This is just Eq. (21.17) with $d/dv$ replaced by $\partial/\partial v$ because $F_0$ now varies slowly in space and time as well as varying with $v$.

Turn, now, from the rapidly varying quantities $F_1$ and $E$ and their Vlasov equation, dispersion relation, and damping rate, to the spatially averaged distribution function $F_0$ and its spatially averaged Vlasov equation (22.4). We shall bring this Vlasov equation's nonlinear term into a more useful form. The key quantity in this nonlinear term is the average of the product of the rapidly varying quantities $F_1$ and $E$. Parseval's theorem permits us to rewrite this as

$$\langle EF_1 \rangle = \frac{1}{L}\int_0^L EF_1 dz = \frac{1}{L}\int_{-\infty}^{\infty} EF_1 dz = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{\tilde{E}^*\tilde{F}_1}{L} \; , \tag{22.15}$$

where in the second step we have used our mathematical idealization that $F_1$ and $E$ vanish outside our averaging box, and the third equality is Parseval's theorem. Inserting Eq. (22.14), we bring this into the form

$$\langle EF_1 \rangle = \frac{e}{m_e}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{\tilde{E}^*\tilde{E}}{L}\frac{i}{\omega - kv}\frac{\partial F_0}{\partial v} \; . \tag{22.16}$$

The quantity $\tilde{E}^*\tilde{E}/L$ is a function of wave number $k$, time $t$, and also the location and size $L$ of the averaging box. In order for Eq. (22.16) to be physically and computationally useful, it is essential that this quantity *not* fluctuate wildly as $k$, $t$, $L$, and the box location are varied. In most circumstances, if the box is chosen to be far larger than $\lambda\!\!\!^{-} = 1/k$, then $\tilde{E}^*\tilde{E}/L$ indeed will not fluctuate wildly. When one develops the quasilinear theory with greater care and rigor than we can do in so short a treatment, one discovers that this non-fluctuation is a consequence of the *Random Phase Approximation* or RPA for short—an approximation which says that the phase of $\tilde{E}$ varies randomly with $k$, $t$, $L$, and the box location on suitably short lengthscales.[1] Like *ergodicity* (Secs. 3.5 and 5.3), although the RPA is often valid, sometimes it can fail. Sometimes there is an organized bunching of the particles in phase space that induces nonrandom phases on the plasma waves. Quasilinear theory requires that RPA be valid and for the the moment we shall assume it so, but in Sec. 22.6 we shall meet an example for which it fails: strong ion-acoustic solitons.

The RPA implies that, as we increase the length $L$ of our averaging box, $\tilde{E}^*\tilde{E}/L$ will approach a well-defined limit. This limit is 1/2 the spectral density $S_E(k)$ of the random process $E(z,t)$ at fixed time $t$; cf. Eq. (5.33). Correspondingly, it is natural to express quasi-linear theory in the language of spectral densities. We shall do so, but with a normalization of the spectral density that is tied to the physical energy density and differs slightly from that used in Chap. 5: In place of $S_E(k)$, we use the *Langmuir-wave spectral energy density*, $\mathcal{E}_k$. We follow plasma physicists' conventions by defining this quantity to include the oscillatory kinetic energy in the electrons, as well as the electrical energy to which it is, on average,

[1] For detailed discussion see Pines, D. and Schrieffer, J. R. 1962, *Phys. Rev.* **125**, 804; also Davidson, R. C. 1972, *Methods in Nonlinear Plasma Theory*, New York: Academic Press.

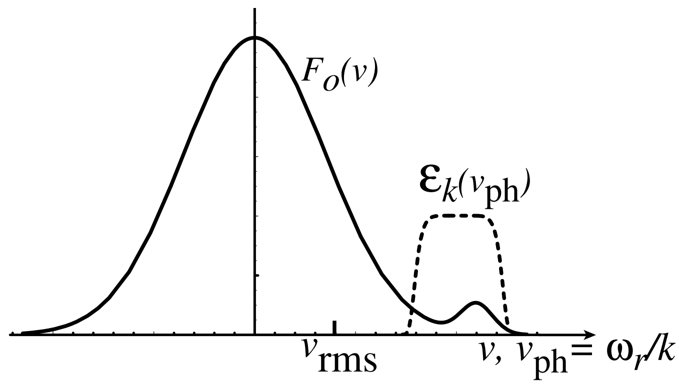


**Fig. 22.1:** The spatially averaged electron velocity distribution $F_o(v)$ (solid curve) and wave energy distribution $\mathcal{E}_k(v_{\rm ph})$ (dotted curve) for the situation treated in Secs. 22.2–22.4.

equal. As in the theory of random processes (Chap. 5) we shall add the energy at $-k$ to that at $+k$, so that all the energy is regarded as residing at positive wave number, and $\int_0^\infty dk\, \mathcal{E}_k$ is the total wave energy per unit volume in the plasma, averaged over length $L$.

Invoking the RPA, we can use Parseval's theorem to compute the electrical energy density

$$\frac{\epsilon_0 \langle E^2 \rangle}{2} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \epsilon_0 \left\langle \frac{\tilde{E}\tilde{E}^*}{2L} \right\rangle = \int_0^\infty \frac{dk}{2\pi} \epsilon_0 \left\langle \frac{\tilde{E}\tilde{E}^*}{L} \right\rangle \,, \tag{22.17}$$

where we have used $\tilde{E}(k)\tilde{E}^*(k) = \tilde{E}(-k)\tilde{E}^*(-k)$. We double this to account for the wave energy in the oscillating electrons and then read off the spectral energy density as the integrand:

$$\mathcal{E}_k = \frac{\epsilon_0 \langle \tilde{E}\tilde{E}^* \rangle}{\pi L} \,. \tag{22.18}$$

This wave energy density can be regarded as a function either of wave number $k$ or wave phase velocity $v_{\rm ph} = \omega_r/k$. It is useful to plot $\mathcal{E}_k(v_{\rm ph})$ on the same graph as the averaged electron velocity distribution $F_o(v)$. Figure 22.1 is such a plot. It shows the physical situation we are considering: approximately thermalized electrons with (possibly) a weak beam (bump) of additional electrons at velocities $v \gg v_{\rm rms}$; and a distribution of Langmuir waves with phase velocities $v_{\rm ph} \gg v_{\rm rms}$.

There is an implicit time dependence associated with the growth or decay of the waves, so that $\mathcal{E}_k \propto e^{2\omega_i t}$. Moreover, since the waves' energy density travels through phase space (physical space and wave vector space) on the same trajectories as a wave packet, i.e. along the geometric-optics rays discussed in Chap. 6

$$\left(\frac{dx_j}{dt}\right)_{\rm wp} = \frac{\partial \omega_r}{\partial k_j} = V_{g\,j} \,, \quad \left(\frac{dk_j}{dt}\right)_{\rm wp} = -\frac{\partial \omega_r}{\partial x_j} \tag{22.19}$$

[Eqs. (6.22) and (6.23)], this growth or decay actually occurs along a wave-packet trajectory (with the averaging box also having to be carried along that trajectory). Thus, the equation of motion for the waves' energy density is

$$\frac{d\mathcal{E}_k}{dt} \equiv \frac{\partial \mathcal{E}_k}{\partial t} + \left(\frac{dz}{dt}\right)_{\rm wp} \frac{\partial \mathcal{E}_k}{\partial z} + \left(\frac{dk}{dt}\right)_{\rm wp} \frac{\partial \mathcal{E}_k}{\partial k} = 2\omega_i \mathcal{E}_k \,. \tag{22.20}$$

Here we have used the fact that our electrostatic waves are presumed to propagate in the $z$ direction, so the only nonzero component of $\mathbf{k}$ is $k_z \equiv k$ and the only nonzero component of the group velocity is $V_{g\,z} = (dz/dt)_{\rm wp} = (\partial\omega_r/\partial k)_z$. For weakly damped, high-phase-speed Langmuir waves, $\omega_r(\mathbf{x}, k)$ is given by Eq. (22.12), with the $\mathbf{x}$-dependence arising from the slowly spatially varying electron density $n(\mathbf{x})$, which induces a slow spatial variation in the plasma frequency $\omega_p = \sqrt{e^2 n/\epsilon_0 m_e}$.

Now, the context in which the quantity $\tilde{E}(k)^*\tilde{E}(k)/L = (\pi L/\epsilon_0)\mathcal{E}_k$ arose was our evaluation of the nonlinear term in the Vlasov equation (22.4) for the electrons' averaged distribution function $F_0$. By inserting (22.16) and (22.18) into (22.4), we bring that nonlinear Vlasov equation into the form

$$\frac{\partial F_0}{\partial t} + v\frac{\partial F_0}{\partial z} = \frac{\partial}{\partial v}\left(D\frac{\partial F_0}{\partial v}\right) , \qquad (22.21)$$

where

$$D(v) = \frac{e^2}{2\epsilon_0 m_e^2}\int_{-\infty}^{\infty} dk\, \mathcal{E}_k \frac{i}{\omega - kv}$$
$$= \frac{e^2}{\epsilon_0 m_e^2}\int_0^{\infty} dk\, \mathcal{E}_k \frac{\omega_i}{(\omega_r - kv)^2 + \omega_i^2} . \qquad (22.22)$$

Here in the second step we have used Eq. (22.10).

Equation (22.21) says that $F_0(v, z, t)$ is transported in physical space with the electron speed $v$ and diffuses in velocity space with the *velocity diffusion coefficient* $D(v)$. Notice that $D(v)$ is manifestly real, and a major contribution to it comes from waves whose phase speeds $\omega_r/k$ nearly match the particle speed $v$, i.e. from *resonant waves*.

The two-lengthscale approximation that underlies quasilinear theory requires that the waves grow or damp on a lengthscale long compared to a wavelength, and equivalently that $|\omega_i|$ be much smaller than $\omega_r$. This allows us, for each $v$, to split the integral in Eq. (22.22) into a piece due to modes that can resonate with electrons of that speed because $\omega_r/k \simeq v$, plus a piece that cannot so resonate. We consider these two pieces in turn.

The resonant piece can be written down, in the limit $|\omega_i| \ll \omega_r$, by approximating the resonance in Eq. (22.22) as a delta function

$$D^{\rm res} \simeq \frac{e^2\pi}{\epsilon_0 m_e^2}\int_0^{\infty} dk\mathcal{E}_k\delta(\omega_r - kv) . \qquad (22.23)$$

In the diffusion equation (22.21) this influences $F_0(v)$ only at velocities $v$ where there reside resonating waves with substantial wave energy, i.e. out on the tail of the electron velocity distribution, under the dotted $\mathcal{E}_k$ curve of Fig. 22.1. We shall refer to electrons in this region as the *resonant electrons*. In Sec. 22.4 below, we will explore the dynamical influence of this resonant diffusion coefficient on the velocity distribution $F_0(v)$ of the resonant electrons.

The vast majority of the electrons reside at velocities $|v| \lesssim v_{\rm rms}$, where there are no waves (because waves there get damped very quickly). For these *non-resonant electrons* the denominator in Eq. (22.22) for the diffusion coefficient is approximately equal to $\omega_r^2 \simeq \omega_p^2 =$

$e^2n/\epsilon_0 m_e$, and correspondingly the diffusion coefficient has the form

$$D^{\rm non-res} \simeq \frac{1}{nm_e}\int_0^\infty \omega_i \mathcal{E}_k dk \ . \tag{22.24}$$

The nonresonant electrons at $v \lesssim v_{\rm rms}$ are the ones that participate in the wave motions and account for the waves' oscillating charge density. The time averaged kinetic energy of these nonresonant electrons thus must include a conserved piece not associated with the waves, plus a wave piece that is equal to the waves' electrical energy and thus to half the waves' total energy, $\frac{1}{2}\int_0^\infty dk\mathcal{E}_k$, and which thus must change at a rate $2\omega_i$ times that energy. Correspondingly, we expect the nonresonant piece of $D(v)$ to produce a change of the time-averaged electron energy given by

$$\frac{\partial U_e}{\partial t} + \frac{\partial \mathcal{F}_{ez}}{\partial z} = \frac{1}{2}\int_0^\infty dk\, 2\omega_i \mathcal{E}_k \ , \tag{22.25}$$

where $\mathcal{F}_{ez}$ is the electron energy flux. Indeed, this is the case; see Ex. 22.1. Because we have already accounted for this electron contribution to the wave energy in our definition of $\mathcal{E}_k$, we shall ignore it henceforth in the evolution of $F_0(v)$, and correspondingly, for weakly damped or growing waves *we shall focus solely on the resonant part of the diffusion coefficient, Eq. (22.23).*

### 22.2.2 Summary of Quasilinear Theory

All the fundamental equations of quasilinear theory are now in hand. They are: (i) The general dispersion relation (21.30) for the waves' frequency $\omega_r(k)$ and growth rate $\omega_i(k)$ [which, for the high-speed Langmuir waves on which we are focusing, reduces to Eqs. (22.12) and (22.13)]; this dispersion relation depends on the electrons' slowly-evolving time-averaged velocity distribution $F_0(v, z, t)$. (ii) The equation of motion (22.20) for the waves' slowly evolving spectral energy density $\mathcal{E}_k(k, z, t)$, in which appear $\omega_r(k)$ and $\omega_i$. (iii) Equation (22.22) or (22.23) for the diffusion coefficient $D(v)$ in terms of $\mathcal{E}_k$. (iv) The diffusive evolution equation (22.21) for the slow evolution of $F_0(v, z, t)$. The fundamental functions in this theory are $\mathcal{E}_k(k, z, t)$ for the waves and $F_0(v, z, t)$ for the electrons.

Quasilinear theory sweeps under the rug and ignores the details of the oscillating electric field $E(z, t)$ and the oscillating part of the distribution function $F_1(v, z, t)$. Those quantities were needed in deriving the quasilinear equations, but they are needed no longer—except, sometimes, as an aid to physically understanding.

### 22.2.3 Conservation Laws

It is instructive to verify that the quasilinear equations enforce the conservation of particles, momentum and energy.

We begin with particles (electrons). The number density of electrons is $n = \int F_0 dv$ and the $z-$*component* of particle flux is $S_z = \int nvdv$ (where here and below all velocity integrals go from $-\infty$ to $+\infty$). Therefore, by integrating the diffusive evolution equation (22.21) for $F_0$ over velocity, we obtain

$$\frac{\partial n}{\partial t} + \frac{\partial S_z}{\partial z} = \int \left( \frac{\partial}{\partial t} F_0 + v \frac{\partial}{\partial z} F_0 \right) dv = \int \frac{\partial}{\partial v} \left( D \frac{\partial F_0}{\partial v} \right) dv = 0 \ , \tag{22.26}$$

which is the law of particle conservation for our one-dimensional situation where there is no dependence of anything on $x$ or $y$.

The $z$ component of electron momentum density is $G_z^e \equiv \int m_e v F_0$ and the $zz$ component of electron momentum flux (stress) is $T_{zz}^e = \int m_e v^2 F_0$; so evaluating the first moment of the evolution equation (22.21) for $F_0$ we obtain

$$\frac{\partial G_z^e}{\partial t} + \frac{\partial T_{zz}^e}{\partial z} = m_e \int v \left( \frac{\partial F_0}{\partial t} + v \frac{\partial F_0}{\partial z} \right) dv = m_e \int v \frac{\partial}{\partial v} \left( D \frac{\partial F_0}{\partial v} \right) dv = -m_e \int D \frac{\partial F_0}{\partial v} dv \ , \tag{22.27}$$

where we have used integration by parts in the last step. The waves influence the momentum of the resonant electrons through the delta-function part of the diffusion coefficient $D$ [Eq. (22.23)], and the momentum of the nonresonant electrons through the remainder of the diffusion coefficient [difference between Eqs. (22.22) and (22.23); cf. Exercise 22.1]. Because we have included the evolving part of the nonresonant electrons' momentum and energy as part of the waves' momentum and energy, we must restrict attention in Eq. (22.27) to the resonant electrons; cf. last sentence in Sec. 22.2.1. We therefore insert the delta-function part of $D$ [Eq. (22.23)] into Eq. (22.27), thereby obtaining:

$$\frac{\partial G_z^e}{\partial t} + \frac{\partial T_{zz}^e}{\partial z} = -\frac{\pi e^2}{\epsilon_0 m_e} \int dv \int dk \mathcal{E}_k \delta(\omega_r - kv) \frac{\partial F_0}{\partial v} = \frac{\pi e^2}{\epsilon_0 m_e} \int \mathcal{E}_k F_0'(\omega_r/k) \frac{dk}{k} \ . \tag{22.28}$$

Here we have exchanged the order of integration, integrated out $v$, and set $F_0'(v) \equiv \partial F_0/\partial v$. Assuming, for definiteness, high-speed Langmuir waves, we can rewrite the last expression in terms of $\omega_i$ with the aid of Eq. (22.13):

$$\begin{aligned} \frac{\partial G_z^e}{\partial t} + \frac{\partial T_{zz}^e}{\partial z} &= -2 \int dk \omega_i \frac{\mathcal{E}_k k}{\omega_r} = -\frac{\partial}{\partial t} \int dk \frac{\mathcal{E}_k k}{\omega_r} - \frac{\partial}{\partial z} \int dk \frac{\mathcal{E}_k k}{\omega_r} \frac{\partial \omega_r}{\partial k} dk \\ &= -\frac{\partial G_z^w}{\partial t} - \frac{\partial T_{zz}^w}{\partial z} \ . \end{aligned} \tag{22.29}$$

The second equality follows from Eq. (22.20). In the last two terms on the first line, $\int dk\, \mathcal{E}_k k/\omega_r = G_z^w$ is the waves' density of $z$-component of momentum (as one can see from the fact that each plasmon carries a momentum $p_z = \hbar k$ and an energy $\hbar \omega_r$; cf. Sec. 22.3 below). Similarly, since the waves' momentum and energy travel with the group velocity $d\omega_r/dk$, $\int dk\, \mathcal{E}_k (k/\omega_r)(\partial \omega_r/\partial k) = T_{zz}^w$ is the waves' flux of momentum. Obviously, Eq. (22.29) represents the conservation of total momentum, that of the resonant electrons plus that of the waves (which includes the evolving part of the nonresonant electron momentum).

Energy conservation can be handled in a similar fashion; see Ex. 22.2.

### 22.2.4 Generalization to Three Dimensions

We have so far restricted our attention to Langmuir waves propagating in one direction, $+\mathbf{e}_z$. The generalization to three dimensions is straightforward: The waves' wave number $k$ gets replaced by the wave vector $\mathbf{k} = k\hat{\mathbf{k}}$, where $\hat{\mathbf{k}}$ is a unit vector in the direction of the phase velocity. The waves' spectral energy density becomes $\mathcal{E}_{\mathbf{k}}$, which depends on $\mathbf{k}$, varies slowly in space $\mathbf{x}$ and time $t$, and is related to the waves' total energy density by

$$U_w = \int \mathcal{E}_{\mathbf{k}} dV_k \ , \tag{22.30}$$

where $dV_k \equiv dk_x dk_y dk_z$ is the volume integral in wave-vector space.

Because the plasma is isotropic, the dispersion relation $\omega(k) = \omega(|\mathbf{k}|)$ has the same form as in the one-dimensional case, and the group and phase velocities point in the same direction $\hat{\mathbf{k}}$: $\mathbf{V}_{\rm ph} = (\omega/k)\hat{\mathbf{k}}$, $\mathbf{V}_g = (d\omega_r/dk)\hat{\mathbf{k}}$. The evolution equation (22.20) for the waves' spectral energy density moving along a ray (a wave-packet trajectory) becomes

$$\frac{d\mathcal{E}_{\mathbf{k}}}{dt} \equiv \frac{\partial \mathcal{E}_k}{\partial t} + \left(\frac{dx_j}{dt}\right)_{\rm wp} \frac{\partial \mathcal{E}_j}{\partial z} + \left(\frac{dk_j}{dt}\right)_{\rm wp} \frac{\partial \mathcal{E}_{\mathbf{k}}}{\partial k_j} = 2\omega_i \mathcal{E}_{\mathbf{k}} \ . \tag{22.31}$$

The equation for the wave-packet trajectory remains (22.19), unchanged. The diffusion coefficient acts only along the direction of the waves; i.e. its $\hat{\mathbf{k}} \otimes \hat{\mathbf{k}}$ component has the same form (22.23) as in the one-dimensional case and components orthogonal to $\hat{\mathbf{k}}$ vanish, so

$$\mathsf{D} = \frac{\pi e^2}{\epsilon_0 m_e^2} \int \mathcal{E}_{\mathbf{k}} \, \hat{\mathbf{k}} \otimes \hat{\mathbf{k}} \, \delta(\omega_r - \mathbf{k} \cdot \mathbf{v}) dV_k \ . \tag{22.32}$$

Because the waves will generally propagate in a variety of directions, the net $\mathsf{D}$ is not unidirectional. This diffusion coefficient enters into the obvious generalization of the evolution equation (22.21) for the averaged distribution function $f_0(v)$—on which we shall suppress the subscript 0 for ease of notation:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} f = \boldsymbol{\nabla}_v \cdot (\mathsf{D} \cdot \boldsymbol{\nabla}_v f) \ . \tag{22.33}$$

Here $\boldsymbol{\nabla}_v$ is the gradient in velocity space, i.e. in index notation $\partial/\partial v_j$.

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**EXERCISES**

**Exercise 22.1** *Problem: Non-resonant Particle Energy in Wave*
Show that the *non-resonant* part of the diffusion coefficient in velocity space, Eq. (22.24), produces a rate of change of electron kinetic energy given by Eq. (22.25).

**Exercise 22.2** *Problem: Energy Conservation*
Show that the quasilinear evolution equations guarantee conservation of total energy, that of the resonant electrons plus that of the waves. Pattern your analysis after that for momentum, Eqs. (22.27)–(22.29).

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# 22.3 Quasilinear Theory in Quantum Mechanical Language

## 22.3.1 Wave-Particle Interactions

The attentive reader, will have noticed a familiar structure to our quasilinear theory. It is reminiscent of the geometric optics formalism that we introduced in Chap. 6. Here as there we can reinterpret the formalism in terms of quanta carried by the waves. At the most fundamental level, we could (second) quantize the field of plasma waves into quanta, (usually called *plasmons*) and describe their creation and annihilation using quantum mechanical transition probabilities. However, there is no need to go through the rigors of the quantization procedure, since the basic concepts of creation, annihilation, and transition probabilities should already be familiar to most readers in the context of photons coupled to atomic systems. Those concepts can be carried over essentially unchanged to plasmons, and by doing so we shall recover our quasilinear theory rewritten in quantum language.

A major role in the quantum theory is played by the *occupation number* for electrostatic wave modes, which are just the single-particle states of the quantum theory. Our electrostatic waves have spin zero, since there is no polarization freedom (the direction of $\mathbf{E}$ is unique: it must point along $\mathbf{k}$). In other words, there is only one polarization state for each $\mathbf{k}$, so the number of modes (i.e. of quantum states) in a volume $dV_x dV_k = dxdydzdk_xdk_ydk_z$ of phase space is $dN_{\rm states} = dV_x dV_k/(2\pi)^3$, and correspondingly the number density of states in phase space is $dN_{\rm states}/dV_x dV_k = 1/(2\pi)^3$; cf. Sec. 2.3. The density of energy in phase space is $\mathcal{E}_{\mathbf{k}}$, and the energy of an individual plasmon is $\hbar\omega_r$, so the number density of plasmons in phase space is $dN/dV_x dV_k = \mathcal{E}_{\mathbf{k}}/\hbar\omega_r$. Therefore, the states' occupation number is given by

$$\begin{aligned}\eta(\mathbf{k},\mathbf{x},t) &= \frac{dN/dV_x dV_k}{dN_{\rm states}/dV_x dV_k} \\ &= \frac{dN}{dV_x\ dV_k/(2\pi)^3} \\ &= \frac{(2\pi)^3\mathcal{E}_{\mathbf{k}}}{\hbar\omega_r}\ . \end{aligned} \tag{22.34}$$

This is actually the *mean occupation number*; the occupation numbers of individual states will fluctuate statistically around this mean. In this chapter (as in most of our treatment of statistical physics in Part 1 of this book), we will not deal with the individual occupation numbers, since quasilinear theory is oblivious to them and deals only with the mean. Thus, without any danger of ambiguity we shall simplify our terminology by suppressing the word "mean".

Equation (22.34) says that $\eta(\mathbf{k},\mathbf{x},t)$ and $\mathcal{E}_{\mathbf{k}}$ are the same quantity, aside from normalization. In the classical formulation of quasilinear theory we use $\mathcal{E}_{\mathbf{k}}$; in the equivalent quantum formulation we use $\eta$. We can think of $\eta$ equally well as a function of the state's wave number $\mathbf{k}$ or of the momentum $\mathbf{p} = \hbar\mathbf{k}$ of the individual plasmons that reside in the state. The motion of these individual plasmons is governed by Hamilton's equations with the Hamiltonian determined by their dispersion relation via $H(\mathbf{p},\mathbf{x},t) = \hbar\omega_r(\mathbf{k} = \mathbf{p}/\hbar,\mathbf{x},t)$; see Sec.

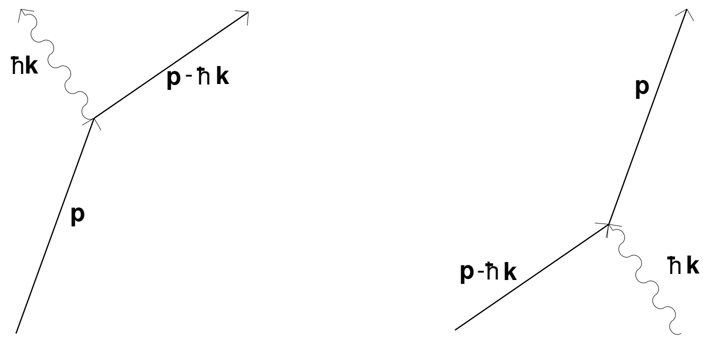


**Fig. 22.2:** Feynman diagrams showing creation and annihilation of a plasmon (with momentum $\hbar\mathbf{k}$) by an electron (with momentum $\mathbf{p}$).

6.3.1. The plasmon trajectories in phase space are, of course, identical to the wave-packet trajectories (the rays) of the classical wave theory.

The second line of Eq. (22.34) allows us to think of $\eta$ as the number density in $\mathbf{x}, \mathbf{k}$ phase space, with the relevant phase space volume renormalized from $dV_k$ to $dV_k/(2\pi)^3 = (dk_x/2\pi)(dk_y/2\pi)(dk_z/2\pi)$. The factors of $2\pi$ appearing here are the same ones as appear in the relationship between a spatial Fourier transform and its inverse [e.g., Eq. (22.8)]. We shall see the quantity $dV_k/(2\pi)^3$ appearing over and over again in the quantum mechanical theory, and it can generally be traced to that Fourier transform relationship.

Resonant interactions between the waves and resonant electrons cause $\mathcal{E}_\mathbf{k}$ to evolve as $d\mathcal{E}_\mathbf{k}/dt = 2\omega_i \mathcal{E}_\mathbf{k}$ [Eq. (22.31)]. Therefore, the plasmon occupation number will also vary as

$$\frac{d\eta}{dt} \equiv \frac{\partial \eta}{\partial t} + \frac{dx_j}{dt} \cdot \frac{\partial \eta}{\partial x_j} + \frac{dp_j}{dt}\frac{\partial \eta}{\partial p_j} = 2\omega_i \eta \ . \tag{22.35}$$

Here the plasmon velocity $dx_j/dt$, as deduced from Hamilton's equations, is equal (of course) to a wave packet's group velocity $V_{g\,j}$, and the force acting on a plasmon, $dp_j/dt$, as deduced from Hamilton's equations, is $\hbar$ times the ray equations' $dk_j/dt = -\partial\omega_r/\partial x_j$.

The fundamental process that we are dealing with in Eq. (22.35) is the creation (or annihilation) of a plasmon by an electron; cf. Fig. 22.2. The kinematics of this process is simple: energy and momentum must be conserved in the interaction. In plasmon creation (left diagram), the plasma gains one quantum of energy $\hbar\omega_r$ so the electron must lose this same energy, $\hbar\omega_r = -\Delta(m_e v^2/2) \simeq -\Delta\mathbf{p}\cdot\mathbf{v}$, where $\Delta\mathbf{p}$ is the electron's change of momentum and $\mathbf{v}$ is its velocity, and in "$\simeq$" we assume that the electron's fractional change of energy is small (an assumption inherent in quasilinear theory). Since the plasma momentum change, $\hbar\mathbf{k}$, is minus the electron momentum change, we conclude that

$$\hbar\omega_r = -\Delta(m_e v^2/2) \simeq -\Delta\mathbf{p}\cdot\mathbf{v} = \hbar\mathbf{k}\cdot\mathbf{v} \ . \tag{22.36}$$

This is just the resonance condition contained in the delta function (22.32). Thus, energy and momentum conservation in the fundamental plasmon creation process imply that the electron producing the plasmon must resonate with the plasmon's mode, i.e. the component of the electron's velocity along $\mathbf{k}$ must be the same as the mode's phase speed, i.e. the electron must "surf" with the wave mode in which it is creating the plasmon, always remaining in the same trough or crest of the mode.

A fundamental quantity in the quantum description is the probability per unit time for an electron with velocity $\mathbf{v}$ to *spontaneously* emit a Langmuir plasmon into a volume $\Delta V_k$ in $\mathbf{k}$-space; i.e. *the number of* $\mathbf{k}$ *plasmons emitted by a single* $\mathbf{v}$ *electron per unit time into the volume* $\Delta V_k$ *centered on some wave vector* $\mathbf{k}$. This probability is expressed in the following form:

$$\frac{dN_{\text{plasmons}}}{dt} \equiv W(\mathbf{v}, \mathbf{k}) \frac{\Delta V_k}{(2\pi)^3} \; . \tag{22.37}$$

In a volume $\Delta V_x$ of physical space and $\Delta V_v$ of electron velocity space, there are $f(\mathbf{v})\Delta V_x \Delta V_v$ electrons; and Eq. (22.37) tells us that these electrons increase the number of plasmons in $\Delta V_x$ and $\Delta V_k$ by

$$\frac{dN_{\text{plasmons}}}{dt} \equiv W(\mathbf{v}, \mathbf{k}) \frac{\Delta V_k}{(2\pi)^3} f(\mathbf{v}) \Delta V_x \Delta V_v \; . \tag{22.38}$$

Dividing by $\Delta V_x$ and by $\Delta V_k/(2\pi)^3$, we obtain for the rate of change of the plasmon occupation number [second line of Eq. (22.34)]

$$\frac{d\eta(\mathbf{k})}{dt} = W(\mathbf{v}, \mathbf{k}) f(\mathbf{v}) \Delta V_v \; . \tag{22.39}$$

Integrating over all of velocity space, we obtain a final expression for the influence of spontaneous plasmon emission on the plasmon occupation number:

$$\frac{d\eta(\mathbf{k})}{dt} = W(\mathbf{v}, \mathbf{k}) f(\mathbf{v}) \Delta V_v \; . \tag{22.40}$$

Our introduction of the factor $(2\pi)^3$ in the definition (22.37) of $W(\mathbf{v}, \mathbf{k})$ was designed to avoid a factor $(2\pi)^3$ in this equation for the evolution of the occupation number.

Below, we shall deduce the fundamental emission rate $W$ for high-speed Langmuir phonons by comparison with our classical formulation of quasilinear theory.

Because the plasmons have spin zero, they obey Bose-Einstein statistics, which means that the rate for *induced* emission of plasmons is larger than that for spontaneous emission by the occupation number $\eta$ of the state that receives the plasmons. Furthermore, the principle of detailed balance (unitarity in quantum mechanical language) tells us that $W$ is also the relevant transition probability for the inverse process of absorption of a plasmon in a transition between the same two electron momentum states (right diagram in Fig. 22.2). This permits us to write down a *master equation* for the evolution of the plasmon occupation number in a homogeneous plasma:

$$\frac{d\eta}{dt} = \int dV_v W(\mathbf{v}, \mathbf{k}) \{ f(\mathbf{v})[1 + \eta(\mathbf{k})] - f(\mathbf{v} - \hbar\mathbf{k}/m_e)\eta(\mathbf{k}) \} \tag{22.41}$$

The first term in the square brackets in the integrand is the contribution from spontaneous emission, the second term is induced emission, and the final term (after the square brackets) is absorption.

The master equation (22.41) is actually the evolution law (22.35) for $\eta$ in disguise, with the e-folding rate $\omega_i$ written in a fundamental quantum mechanical form. To make contact

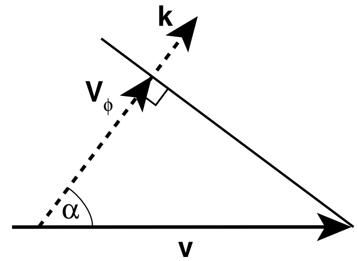


**Fig. 22.3:** Emission geometry for Cerenkov emission. An electron at $P$ moving with speed $v$ emits waves with phase speed $V_{\rm ph} < v$ along a direction that make an angle $\alpha = \cos^{-1}(V_{\rm ph}/v)$ with the direction of the electron's motion.

with Eq. (22.35), we first notice that in our our classical development of quasilinear theory we neglected spontaneous emission, so we drop it from Eq. (22.41). In the absorption term, the momentum of the plasmon is so much smaller than the electron momentum that we can make a Taylor expansion

$$f(\mathbf{v} - \hbar\mathbf{k}/m_e) \simeq f(\mathbf{v}) - (\hbar/m_e)(\mathbf{k}\cdot\boldsymbol{\nabla}_v)f \ . \tag{22.42}$$

Inserting this into Eq. (22.41) and removing the spontaneous-emission term, we obtain

$$\frac{d\eta}{dt} \simeq \eta \int W \frac{\hbar}{m_e}(\mathbf{k}\cdot\boldsymbol{\nabla}_v) f dV_v \ . \tag{22.43}$$

For comparison, Eq. (22.35), with $\omega_i$ given by the classical high-speed Langmuir relation (22.13) and converted to 3-dimensional notation, says that

$$\frac{d\eta}{dt} = \eta \int \frac{\pi e^2\omega_r}{\epsilon_0 k^2 m_e}\delta(\omega_r - \mathbf{k}\cdot\mathbf{v})\mathbf{k}\cdot\boldsymbol{\nabla}_v f dV_v \ . \tag{22.44}$$

Comparing Eqs. (22.43) and (22.44), we see that the fundamental quantum emission rate for plasmons is

$$W = \frac{\pi e^2\omega_r}{\epsilon_0 k^2\hbar}\delta(\omega_r - \mathbf{k}\cdot\mathbf{v}) \ . \tag{22.45}$$

Note that this emission rate is inversely proportional to $\hbar$ and is therefore a very large number under the classical conditions of our quasilinear theory.

This computation has shown that the classical absorption rate $-\omega_i$ is the difference between the quantum mechanical absorption rate and induced emission rate. Under normal conditions, when $\mathbf{k}\cdot\boldsymbol{\nabla}_v f < 0$ ($\partial F_0/\partial v < 0$ in one dimensional language), the absorption dominates over emission, so the absorption rate $-\omega_i$ is positive, describing Landau damping. However (as we saw in Chap. 21), when this inequality is reversed, there can be wave growth (subject of course to there being a suitable mode into which the plasmons can be emitted, as guaranteed when the Penrose criterion is fulfilled).

Although spontaneous emission was absent from our classical development of quasilinear theory, it nevertheless can be a classical process and therefore must be added to the quasilinear formalism. Classically or quantum mechanically, the spontaneous emission is a form of *Cerenkov radiation*, since (as for Cerenkov light emitted by electrons moving through a dielectric medium), the plasmons are produced when an electron moves through the plasma faster than the waves' phase velocity. More specifically, only when $v > V_{\rm ph} = \omega_r/k$ can there be an angle $\alpha$ of $\mathbf{k}$ relative to $\mathbf{v}$ along which the resonance condition is satisfied, $\mathbf{v}\cdot\hat{\mathbf{k}} = v\cos\alpha = \omega_r/k$. The plasmons are emitted at this angle $\alpha$ to the electron's direction of motion; cf. Fig. 22.3.

The spontaneous Cerenkov emission rate

$$\left(\frac{d\eta}{dt}\right)_s = \int W(\mathbf{v},\mathbf{k}) f(\mathbf{v}) dV_v \; . \tag{22.46}$$

[Eq. (22.40) integrated over electron velocity] takes the following form when we use the Langmuir expression (22.45) for $W$:

$$\left(\frac{d\eta}{dt}\right)_s = \frac{\pi e^2}{\epsilon_0 \hbar}\int \frac{\omega_r}{k^2} f(\mathbf{v})\delta(\omega_r - \mathbf{k}\cdot\mathbf{v}) dV_v \; . \tag{22.47}$$

Translated into classical language via Eq. (22.34), this rate is

$$\left(\frac{d\mathcal{E}_{\mathbf{k}}}{dt}\right)_s = \frac{e^2}{8\pi^2\epsilon_0}\int \frac{\omega_r^2}{k^2} f(\mathbf{v})\delta(\omega_r - \mathbf{k}\cdot\mathbf{v}) dV_v \; . \tag{22.48}$$

Note that Planck's constant is absent from the classical expression, but present in the quantum one.

In the above analysis we computed the fundamental emission rate $W$ by comparing the quantum induced-emission rate minus absorption rate with the classical growth rate for plasma energy. An alternative route to Eq. (22.45) for $W$ would have been to use classical plasma considerations to compute the classical Cerenkov emission rate (22.48), then convert to quantum language using $\eta = (2\pi)^3\mathcal{E}_{\mathbf{k}}/\hbar\omega_r$ thereby obtaining Eq. (22.47), then compare with the fundamental formula (22.46).

By comparing Eqs. (22.44) and (22.47) and assuming a thermal (Maxwell) distribution for the electron velocities, we see that the spontaneous Cerenkov emission is ignorable in comparison with Landau damping when the electron temperature is smaller than $\eta\hbar\omega/k_B$. Sometimes it is convenient to define a classical brightness temperature $T_B(\mathbf{k})$ for the plasma waves given implicitly by $\eta(\mathbf{k}) = (e^{\hbar\omega/k_B T_B(\mathbf{k})} - 1)^{-1} \sim k_B T_B(\mathbf{k})/\hbar\omega$. In this language, spontaneous emission of plasmons with wave vector $\mathbf{k}$ is generally ignorable when the wave brightness temperature exceeds the electron kinetic temperature—as one might expect on thermodynamic grounds. In a plasma in strict thermal equilibrium, we expect Cerenkov emission to be balanced by Landau damping, so as to maintain a thermal distribution of Langmuir waves with a temperature equal to that of the electrons, $T_B(\mathbf{k}) = T_e$ for all $\mathbf{k}$.

Turn, now, from the evolution of the plasmon distribution $\eta(\mathbf{k},\mathbf{x},t)$ to that of the particle distribution $f(\mathbf{v},\mathbf{x},t)$. Classically, $f$ evolves via the velocity-space diffusion equation (22.33). We shall write down a fundamental quantum mechanical evolution equation (the "kinetic equation") which appears at first sight to differ remarkably from (22.33), but then shall recover (22.33) in the classical limit.

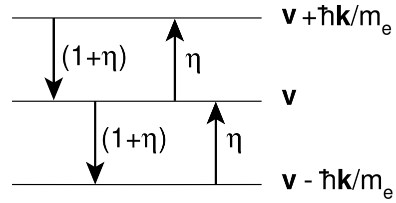


**Fig. 22.4:** Three level system for understanding the electron kinetic equation.

To derive the electron kinetic equation, we must consider three electron velocity states, $\mathbf{v}$ and $\mathbf{v} \pm \hbar\mathbf{k}/m_e$; Fig. 22.4. Momentum conservation says that an electron can move between these states by emission or absorption of a plasmon with wave vector $\mathbf{k}$. The fundamental probability for these transitions is the same one, $W$, as for plasmon emission, since these transitions are merely plasmon emission as seen from the electron viewpoint. Therefore, the electron kinetic equation must take the form

$$\begin{aligned} \frac{df(\mathbf{v})}{dt} = \int \frac{dV_k}{(2\pi)^3} \{ &(1+\eta)[W(\mathbf{v}+\hbar\mathbf{k}/m_e, \mathbf{k}) f(\mathbf{v}+\hbar\mathbf{k}/m_e) - W(\mathbf{v},\mathbf{k}) f(\mathbf{v})] \\ &-\eta[W(\mathbf{v}+\hbar\mathbf{k}/m_e, \mathbf{k}) f(\mathbf{v}) - W(\mathbf{v},\mathbf{k}) f(\mathbf{v}-\hbar\mathbf{k}/m_e)]\} \; . \end{aligned} \tag{22.49}$$

The four terms can be understood by inspection of Fig. 22.4. The two downward transitions in that diagram entail plasmon emission and thus are weighted by $(1+\eta)$, where the 1 is the spontaneous contribution and the $\eta$ the induced emission. In the first of these $(1+\eta)$ terms in Eq. (22.49), the $\mathbf{v}$ electron state gets augmented so the sign is positive; in the second it gets depleted so the sign is negative. The two upward transitions entail plasmon absorption and thus are weighted by $\eta$, and the sign in Eq. (22.49) is plus when the final electron state has velocity $\mathbf{v}$, and minus when the initial state is $\mathbf{v}$.

In the domain of classical quasilinear theory, the momentum of each emitted or absorbed plasmon must be small compared to that of the electron, so we can expand the terms in Eq. (22.49) in powers of $\hbar\mathbf{k}/m_e$. Carrying out that expansion to second order and retaining those terms that are independent of $\hbar$ and therefore classical, we obtain the *quasilinear electron kinetic equation*:

$$\frac{df}{dt} = \boldsymbol{\nabla}_v \cdot [A(\mathbf{v}) f + \mathsf{D}(\mathbf{v}) \cdot \boldsymbol{\nabla}_v f] \; , \tag{22.50}$$

where

$$\begin{aligned} \mathbf{A}(\mathbf{v}) &= \int \frac{dV_k}{(2\pi)^3} \frac{W(\mathbf{v},\mathbf{k})\hbar\mathbf{k}}{m_e} \; , \\ \mathsf{D}(\mathbf{v}) &= \int \frac{dV_k}{(2\pi)^3} \frac{\eta(\mathbf{k}) W(\mathbf{v},\mathbf{k}) \hbar\mathbf{k} \otimes \hbar\mathbf{k}}{m_e^2} \; . \end{aligned} \tag{22.51}$$

The kinetic equation (22.50) is of Fokker-Planck form [Sec. 5.6; Eq. (5.123)]. The quantity $-\mathbf{A}$ is a resistive Fokker-Planck coefficient associated with spontaneous emission; and $\mathsf{D}$, which we can reexpress in the notation of Eq. (5.128) as

$$\mathsf{D} = \left\langle \frac{\Delta\mathbf{v} \otimes \Delta\mathbf{v}}{\Delta t} \right\rangle \tag{22.52}$$

with $\Delta\mathbf{v} = -\hbar\mathbf{k}/m_e$, is the combined resistive-diffusive coefficient that arises when electron recoil can be ignored; it is associated with plasmon absorption and induced emission.

In our classical quasilinear analysis we ignored spontaneous emission and thus had no resistive term in the evolution equation (22.33) for $f$. We can recover that evolution equation and its associated $\mathsf{D}$ by dropping the resistive term from the quantum kinetic equation (22.50) and inserting expression (22.45) for $W$ into Eq. (22.51) for the quantum $\mathsf{D}$. The results agree with the classical equations (22.32) and (22.33).

Let us return briefly to Cerenkov emission by an electron or ion. In the presence of a background magnetic field $\mathbf{B}$, the resonance condition for Cerenkov emission must be modified. Only the momentum parallel to the magnetostatic field need be conserved, not the total vectorial momentum. The unbalanced components of momentum perpendicular to $\mathbf{B}$ are compensated by a reaction force from $\mathbf{B}$ itself and thence by the steady currents that produce $\mathbf{B}$; and correspondingly the unbalanced perpendicular momentum ultimately does work on those currents. In this situation, one can show that the Cerenkov resonance condition $\omega_r - \mathbf{k}\cdot\mathbf{v} = 0$ is modified to

$$\omega_r - \mathbf{k}_{\|}\cdot\mathbf{v}_{\|} = 0\ , \tag{22.53}$$

where $||$ means the component parallel to $\mathbf{B}$. If we allow for the electron gyrational motion as well, then some number of gyrational quanta can be fed into the emitted plasmons, so Eq. (22.53) gets modified to read

$$\omega_r - \mathbf{k}_{\|}\cdot\mathbf{v}_{\|} = n\omega_{ce} \tag{22.54}$$

where $n$ is an integer. For nonrelativistic electrons, the strongest resonance is for $n = 1$,

### 22.3.2 The relationship between classical and quantum mechanical formalisms in plasma physics

We have demonstrated how the structure of the classical quasilinear equations is mandated by quantum mechanics. In developing the quantum equations, we had to rely on one classical calculation, that which gave us the emission rate $W$. However, even this was not strictly necessary, since with significant additional effort we could have have calculated the relevant quantum mechanical matrix elements and then computed $W$ directly from Fermi's golden rule. This has to be the case, because quantum mechanics is the fundamental physical theory and thus must be applicable to plasma excitations just as it is applicable to atoms. Of course, if we are only interested in classical processes, as is usually the case in plasma physics, then we end up taking the limit $\hbar \to 0$ in all observable quantities and the classical rate is all we need.

This raises an important point of principle. Should we perform our calculations classically or quantum mechanically? The best answer is to be pragmatic. Many calculations in nonlinear plasma theory are so long and arduous that we need all the help we can get to complete them. We therefore combine both classical and quantum considerations (confident that both must be correct throughout their overlapping domain of applicability), in whatever proportion minimises our computational effort.

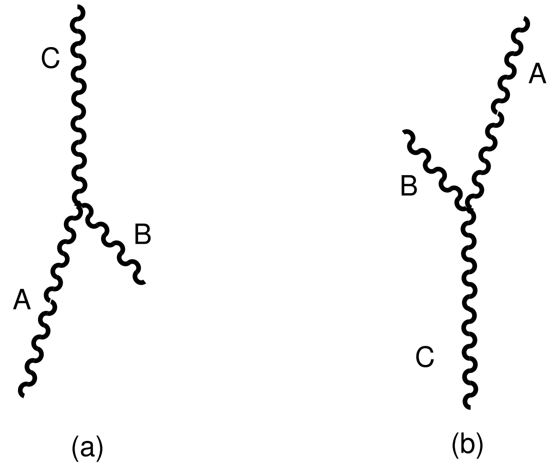


**Fig. 22.5:** (a) A three-wave process in which two plasmons A and B interact nonlinearly to create a third plasmon C. Conserving energy and linear momentum we obtain $\omega_C = \omega_A + \omega_B$ and $\mathbf{k}_C = \mathbf{k}_A + \mathbf{k}_B$. For example A and C might be transverse electromagnetic plasmons satisfying the dispersion relation (20.22) and B might be a longitudinal plasmon (Langmuire or ion acoustic); or A and C might be Langmuir plasmons and B might be an ion acoustic plasmon — the case treated in the text and in Exs. 22.5 and 22.6. (b) The time-reverse three-wave process in which plasmon C generates plasmon B by an analog of Cerenkov emission, and while doing so recoils into plasmon state A.

### 22.3.3 Three-Wave Mixing

We have discussed plasmon emission and absorption both classically and quantum mechanically. Our classical and quantum formalisms can be generalized straightforwardly to encompass other nonlinear processes.

Among the most important other processes are three-wave interactions (in which two waves coalesce to form a third wave or one wave splits up into two) and scattering processes, in which waves are scattered off particles without creating or destroying plasmons. In this section we shall focus on three-wave mixing. We shall present the main ideas in the text but shall leave most of the details to Exs. 22.5 and 22.6.

In three-wave mixing, where waves $A$ and $B$ combine to create a wave $C$ [Fig. 22.5(a)], the equation for the growth of the amplitude of wave $C$ will contain nonlinear driving terms that combine the harmonic oscillations of waves $A$ and $B$, i.e. driving terms proportional to $\exp[i(\mathbf{k}_A \cdot \mathbf{x} - \omega_A t)] \exp[i(\mathbf{k}_B \cdot \mathbf{x} - \omega_B t)]$. In order for wave C to build up coherently over many oscillation periods, it is necessary that the spacetime dependence of these driving terms be the same as that of wave C, $\exp[i(\mathbf{k}_C \cdot \mathbf{x} - \omega_C t)]$; i.e. it is necessary that

$$\mathbf{k}_C = \mathbf{k}_A + \mathbf{k}_B \ , \quad \omega_C = \omega_A + \omega_B \ . \tag{22.55}$$

Quantum mechanically, this can be recognized as momentum and energy conservation for the waves' plasmons. We have met three-wave mixing previously, for electromagnetic waves in a nonlinear dielectric crystal (Sec. 9.5). There, as here, the conservation law (22.55) was necessary in order for the mixing to proceed; see Eq. (9.24) and associated discussion.

When the three waves are all electrostatic, the three-wave mixing arises from the nonlinear term (22.6) in the rapidly varying part (wave part) of the Vlasov equation, which we discarded in our quasilinear analysis. Generalized to three dimensions with this term treated as a driver, that Vlasov equation takes the form

$$\frac{\partial f_1}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} f_1 - \frac{e}{m_e}\mathbf{E}\cdot\boldsymbol{\nabla}_v f_0 = \frac{e}{m_e}\left(\mathbf{E}\cdot\boldsymbol{\nabla}_v f_1 - \langle\mathbf{E}\cdot\boldsymbol{\nabla}_v f_1\rangle\right) . \tag{22.56}$$

In the driving term (right hand side of this equation), $\mathbf{E}$ could be the electric field of wave A and $f_1$ could be the perturbed velocity distribution of wave B or vice versa, and the $E$ and $f_1$ terms on the left side could be those of wave C. If the wave vectors and frequencies are related by (22.55), then via this equation waves A and B will coherently generate wave C.

The dispersion relations for Langmuir and ion acoustic waves permit the conservation law (22.55) to be satisfied if A is Langmuir so $\omega_A \sim \omega_{pe}$, B is ion acoustic so $\omega_B \lesssim \omega_{pp} \ll \omega_A$, and C is Langmuir. By working out the detailed consequences of the driving term (22.56) in the quasilinear formalism and comparing with the quantum equations for 3-wave mixing (Ex. 22.5), one can deduce the fundamental rate for the process $\mathrm{A} + \mathrm{B} \to \mathrm{C}$ [Fig. 22.5(a)]. Detailed balance (unitarity) guarantees that the time reversed process $\mathrm{C} \to \mathrm{A} + \mathrm{B}$ [Fig. 22.5(b)] will have identically the same fundamental rate. This time-reversed process has a physical interpretation analogous to the the emission of a Cerenkov plasmon by a high-speed, resonant electron: C is a "high-energy" Langmuir plasmon ($\omega_C \sim \omega_{pe}$) that can be thought of as Cerenkov-emitting a "low-energy" ion acoustic plasmon ($\omega_B \lesssim \omega_{pp} \ll \omega_A$) and in the process recoiling slightly into Langmuir state A. The fundamental rate that one obtains for this wave-wave Cerenkov process and its time reversal, when the plasma's electrons are thermalized at temperature $T_e$, is [Ex. 22.5[2]].

$$W_{\mathrm{AB}\leftrightarrow\mathrm{C}} = R_{AB\to C}(\mathbf{k}_A, \mathbf{k}_{\mathrm{ia}}, \mathbf{k}_C)\delta(\mathbf{k}_A + \mathbf{k}_{\mathrm{ia}} - \mathbf{k}_{\mathrm{C}})\delta(\omega_A + \omega_{\mathrm{ia}} - \omega_{\mathrm{C}}) . \tag{22.57}$$

where

$$R_{AB\to C}(\mathbf{k}_A, \mathbf{k}_{\mathrm{ia}}, \mathbf{k}_C) = \frac{8\pi^5\hbar e^2 (m_p/m_e)\omega_B^3}{(k_B T_e)^2 k_{\mathrm{ia}}^2}(\hat{\mathbf{k}}_A\cdot\hat{\mathbf{k}}_C)^2 . \tag{22.58}$$

[Here we use the subscript "ia" for the ion acoustic plasmon (plasmon B) to avoid confusion with Boltzmann's constant $k_B$.]

This is the analog of the rate (22.45) for Cerenkov emission by an electron: The ion-acoustic occupation number will evolve via an evolution law analogous to (22.41) with this rate replacing $W$ on the right hand side, $\eta$ replaced by the ion acoustic occupation number $\eta_{\mathrm{ia}}$, and the electron distribution replaced by a product of A-mode and C-mode Langmuir occupation numbers; see Ex. 22.5. Moreover, there will be a similar evolution law for the Langmuir occupation number, involving the same fundamental rate (22.57); Ex. 22.6.

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## EXERCISES

[2]See also Eq. (A.3.12) of Tsytovich, V. N. 1970. *Nonlinear Effects in Plasma*, New York: Plenum. The rates for many other wave-wave mixing processes are worked out in this book, but beware: it contains a large number of typographical errors.

**Exercise 22.3** *Problem: Cerenkov power in electrostatic waves*
Show that the Langmuir wave power radiated by an electron moving with speed $v$ in a plasma with plasma frequency $\omega_p$ is given by

$$P = \frac{e^2\omega_p^2}{4\pi\epsilon_0 v}\ln\left(\frac{k_{\max}v}{\omega_p}\right) \ , \tag{22.59}$$

where $k_{\rm max}$ is the largest wave number at which the waves can propagate. (For larger $k$ the waves are strongly Landau damped.)

**Exercise 22.4** *Derivation: Electron Fokker-Planck Equation*
Fill in the missing details in the derivation of the electron Fokker-Planck equation (22.50)

**Exercise 22.5** *Example and Challenge:Three-Wave Mixing — Ion-Acoustic Evolution*
Consider the three-wave processes shown in Fig. 22.5, with $A$ and $C$ being Langmuir plasmons and $B$ an ion acoustic plasmon and with the fundamental rate being given by Eqs. (22.57) and (22.58).

(a) By summing the rates of forward and backward reactions [diagrams (a) and (b)], show that the occupation number for the ion acoustic plasmons satisfies the kinetic equation

$$\frac{d\eta_B}{dt} = \int W_{AB\leftrightarrow C}[(1+\eta_A+\eta_B)\eta_C - \eta_A\eta_C)]\frac{dV_{k_A}}{(2\pi)^3}\frac{dV_{k_C}}{(2\pi)^3} \ . \tag{22.60}$$

[*Hints:* (i) The rate for A + B → C [Fig. 22.5(a)] will be proportional to $(\eta_C+1)\eta_A\eta_B$; why? (ii) When you sum the rates for the two diagrams, (a) and (b), the terms involving $\eta_A\eta_B\eta_c$ should cancel.]

(b) The ion acoustic plasmons have far lower frequencies than the Langmuir plasmons, so $\omega_B \ll \omega_A \simeq \omega_C$. Assume that they also have far lower wave numbers, $|\mathbf{k}_B| \ll |\mathbf{k}_A| \simeq |\mathbf{k}_C|$. Assume further (as will typically be the case) that the ion acoustic plasmons, because of their tiny individual energies, have far larger occupation numbers than the Langmuir plasmons so $\eta_B \gg \eta_A \sim \eta_C$. Using these approximations, show that the evolution law (22.60) for the ion acoustic waves reduces to the form

$$\frac{d\eta_{\rm ia}(\mathbf{k})}{dt} = \eta_{\rm ia}(\mathbf{k})\int R_{AB\leftrightarrow C}(\mathbf{k}'-\mathbf{k},\mathbf{k},\mathbf{k}')\delta[\omega_{\rm ia}(\mathbf{k}) - \mathbf{k}\cdot\mathbf{V}_{g\,L}(\mathbf{k}')]\mathbf{k}\cdot\boldsymbol{\nabla}_{\mathbf{k}'}\eta_{\rm L}(\mathbf{k}')\frac{dV_k'}{(2\pi)^6} \ , \tag{22.61}$$

where $\eta_L$ is the Langmuir (waves A and C) occupation number, $\mathbf{V}_{g\,L}$ is the Langmuir group velocity, and $R_{C\leftrightarrow BA}$ is the fundamental rate (22.58).

(c) Notice the strong similarities between the evolution equation (22.61) for the ion acoustic plasmons that are Cerenkov-emitted and absorbed by Langmuir plasmons, and the evolution equation (22.44) for Langmuir plasmons Cerenkov-emitted and absorbed by fast electrons! Discuss the similarities and the physical reasons for them.

(d) Carry out an explicit classical calculation of the nonlinear interaction between Langmuir waves with wave vectors $\mathbf{k}_A$ and $\mathbf{k}_C$ to produce ion-acoustic waves with wave vector $\mathbf{k}_B \equiv \mathbf{k}_{\rm ia} = \mathbf{k}_C - \mathbf{k}_A$. Base your calculation on the nonlinear Vlasov equation (22.56) and [for use in relating $\mathbf{E}$ and $f_1$ in the nonlinear term] the 3-dimensional analog of Eq. (22.14). Assume a spatially-independent Maxwellian averaged electron velocity distribution $f_0$ with temperature $T_e$ (so $\boldsymbol{\nabla} f_0 = 0$). From your result compute, in the random phase approximation, the evolution of the ion-acoustic energy density $\mathcal{E}_{\mathbf{k}}$ and thence the evolution of the occupation number $\eta(\mathbf{k})$. Bring that evolution equation into the functional form (22.61). By comparing quantitatively with Eq. (22.61), read off the fundamental rate $R_{C\leftrightarrow BA}$. Your result should be the rate in Eq. (22.58).

**Exercise 22.6** *Example and Challenge:Three-Wave Mixing — Langmuir Evolution*
Continuing the analysis of the preceding excercise:

(a) Derive the kinetic equation for the Langmuir occupation number. [*Hint:* You will have to sum over four Feynman diagrams, corresponding to the mode of interest playing the role of A and then the role of C in each of the two diagrams in Fig. 22.5.]

(b) Using the approximations outlined in part (c), show that the Langmuir occupation number evolves in accord with the diffusion equation

$$\frac{d\eta_L(\mathbf{k}')}{dt} = \boldsymbol{\nabla}_{\mathbf{k}'} \cdot [\mathsf{D}(\mathbf{k}') \cdot \boldsymbol{\nabla}_{\mathbf{k}'}\eta_L(\mathbf{k}')] \;, \tag{22.62}$$

where the diffusion coefficient is given by the following integral over the ion acoustic wave distribution

$$\mathsf{D}(\mathbf{k}') = \int \eta_{\rm ia}(\mathbf{k})\; \mathbf{k}\otimes\mathbf{k}\; R_{C\leftrightarrow BA}(\mathbf{k}-\mathbf{k}', \mathbf{k}', \mathbf{k})\; \delta[\omega_{\rm ia}(\mathbf{k}) - \mathbf{k}\cdot\mathbf{V}_{g\,L}(\mathbf{k}')] \frac{dV'_k}{(2\pi)^6} \;. \tag{22.63}$$

(c) Discuss the strong similarity between this evolution law for resonant Langmuir plasmons interacting with ion acoustic waves, and the one (22.32), (22.33) for resonant electrons interacting with Langmuir waves. Why are they so similar?

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## 22.4 Quasilinear Evolution of Unstable Distribution Functions: The Bump in Tail

A quite common occurence in plasmas arises when a weak beam of electrons passes through a stable Maxwellian plasma with speed $v_b$ large compared with the thermal width of the background plasma $\sigma_e$. When the velocity width of the beam $\sigma_b$ is small compared with $v_b$, the distribution is known as a *bump in tail* distribution; see Fig. 22.6 below. In this section we shall explore the stability and nonlinear evolution of such a distribution.

Let us deal with the simple case of a one dimensional electron distribution function $F_0(v)$ and approximate the beam by a Maxwellian

$$F_b(v) = \frac{n_b}{(2\pi)^{1/2}\sigma_b} e^{-(v-v_b)^2/2\sigma^2} \ , \tag{22.64}$$

where $n_b$ is the beam electron density. For simplicity we shall treat the protons as a uniform neutralizing background.

Now, let us suppose that at time $t = 0$, the beam is established and the Langmuir wave energy density $\mathcal{E}_k$ is very small. The waves will grow fastest when the waves' phase velocity $V_{\rm ph} = \omega_r/k$ resides where the slope of the distribution function is most positive, i.e. when $V_{\rm ph} = v_b - \sigma_b$. The associated maximum growth rate as computed from Eq. (22.13) is

$$\omega_{i\max} = \left(\frac{\pi}{8e}\right)^{1/2} \left(\frac{v_b}{\sigma_b}\right)^2 \left(\frac{n_b}{n_e}\right) \omega_p \ , \tag{22.65}$$

where $e = 2.72...$ is not the electron charge. Now modes will grow over a range of wave phase velocities $\Delta V_{\rm ph} \sim \sigma_b$. By using the Bohm-Gross dispersion relation (20.41) in the form

$$\omega = \omega_p (1 - 3\sigma_e^2/V_{\rm ph}^2)^{-1/2} \ , \tag{22.66}$$

we find that the bandwidth of the growing modes is given roughly by

$$\Delta\omega = K\omega_p \frac{\sigma_b}{v_b} \ , \tag{22.67}$$

where $K = 3(\sigma_e/v_b)^2[1 - 3(\sigma_e/v_b)^2]^{-3/2}$ is a constant $\gtrsim 0.1$ typically. Combining Eqs. (22.66), (22.67) we obtain

$$\frac{\omega_{i\max}}{\Delta\omega} \sim \left(\frac{\pi}{8eK^2}\right)^{1/2} \left(\frac{v_b}{\sigma_b}\right)^3 \left(\frac{n_b}{n_e}\right) \ . \tag{22.68}$$

Dropping constants of order unity, we conclude that the growth time for the waves $\sim (\omega_{i\max})^{-1}$ is long compared with the coherence time $\sim (\Delta\omega)^{-1}$ provided that

$$\sigma_b \gtrsim \left(\frac{n_b}{n_e}\right)^{1/3} v_b \ . \tag{22.69}$$

When inequality (22.69) is satisfied the waves will take several coherence times to grow and so we expect that no permanent phase relations will be established in the electric field and that quasilinear theory is an appropriate tool. However, when this inequality is reversed, the instability resembles more the two stream instability of Chap. 20 and the growth is so rapid as to imprint special phase relations on the waves, so the random phase approximation fails and quasilinear theory is invalid.

Restricting ourselves to slow growth, we shall use the quasilinear theory to explore the evolution of the wave and particle distributions. We can associate the wave energy density $\mathcal{E}_k$ not just with a given value of $k$ but with a corresponding value of $V_{\rm ph} = \omega_r/k$, and thence

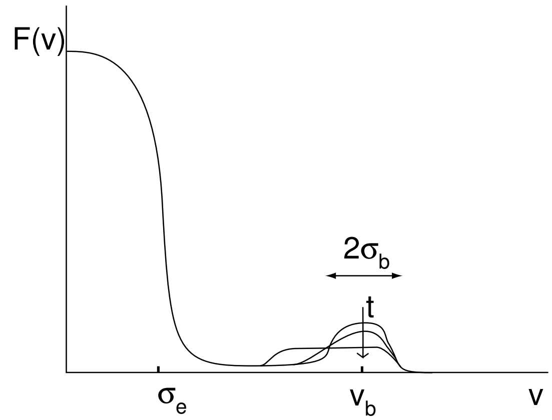


**Fig. 22.6:** Evolution of the one-dimensional electron distribution function from a "bump on tail" shape to a flat distribution function, due to the growth and scattering of electrostatic waves.

with the velocities $v = V_{\rm ph}$ of electrons that resonate with the waves. Using Eq. (22.23) for the velocity diffusion coefficent and Eq. (22.13) for the associated wave growth rate, we can then write the temporal evolution equations for the electron distribution function $F_0(v,t)$ and the wave energy density $\mathcal{E}_k(v,t)$ as

$$\frac{\partial F_0}{\partial t} = \frac{\pi e^2}{m_e^2 \epsilon_0} \frac{\partial}{\partial v} \left( \frac{\mathcal{E}_k}{v} \frac{\partial F_0}{\partial v} \right) ,$$
$$\frac{\partial \mathcal{E}_k}{\partial t} = \frac{\pi e^2}{m_e \epsilon_0 \omega_p} v^2 \mathcal{E}_k \frac{\partial F_0}{\partial v} . \tag{22.70}$$

Here for simplicity we have assumed a spatially homogeneous distribution of particles and waves so $d/dt \to \partial/\partial t$.

This pair of nonlinear equations must be solved numerically, but their qualitative behavior can be understood analytically without much effort; see Fig. 22.6. Waves resonant with the rising part of the electron distribution function at first will grow exponentially, causing the particles to diffuse and flatten the slope of $f$ and thereby reduce the wave growth rate. Ultimately, the slope $\partial F_0/\partial f$ will diminish to zero and the wave energy density will become constant, with its integral, by energy conservation [Ex. 22.2], equal to the total kinetic energy lost by the beam. In this way we see that a velocity space irregularity in the distribution function leads to the growth of electrostatic waves which can react back on the particles in such a way as to saturate the instability. The net result is a beam of particles with a much broadened width propagating through the plasma. The waves will ultimately damp through three-wave processes or other damping mechanisms, sending their energy ultimately into heat.

### 22.4.1 Instability of Streaming Cosmic Rays

For a simple illustration of this general type of instability we return to the issue of the isotropization of Galactic cosmic rays, which we introduced in Sec. 18.7. We argued there that cosmic rays propagating through the interstellar medium are effectively scattered by hydromagnetic Alfvén waves. We did not explain where these Alfvén waves originated. It now seems likely that much of the time these waves are generated by the cosmic rays themselves.

Suppose that we have a beam of cosmic rays propagating through the interstellar gas at high speed. The interstellar gas is magnetized, which allows many more wave modes to propagate than in the unmagnetized case. It turns out that the particle distribution is unstable to the growth of Alfvén waves satisfying the resonance condition (22.54), modified to take account for the fact that we are dealing with mildly relativistic protons rather than non-relativistic electrons:

$$\omega - \mathbf{k}_\parallel \cdot \mathbf{v}_\parallel = \frac{\omega_{cp}}{\gamma} \ . \tag{22.71}$$

Here $\gamma$ is the Lorentz factor of the protons, and we assume that $n = 1$. As the cosmic rays travel much faster than the waves, the effective resonance condition is that the wavelength of the Alfvén wave match the particle gyro radius. The growth rate of these waves can be studied using a kinetic theory analogous to that which we have just developed for Langmuir waves.[3] Dropping factors of order unity, it is given approximately by

$$\omega_i \simeq \left(\frac{n_{cr}}{n_p}\right) \omega_{cp} \left(\frac{u_{cr}}{a} - 1\right) \ , \tag{22.72}$$

where $n_{cr}$ is the number density of cosmic rays, $n_p$ is the number density of thermal protons in the background plasma, $u_{cr}$ is the mean speed of the cosmic ray protons through the background plasma and $a$ is the Alfvén speed. So if the particles have a mean speed in excess of the Alfvén speed, the waves will grow, exponentially at first. It is observed that the energy density of cosmic rays builds up until it is roughly comparable with that of the thermal plasma. As more cosmic rays are produced, they will escape from the Galaxy at a sufficient rate to maintain this balance. Therefore, in a steady state, the ratio of the number density of cosmic rays to the thermal proton density is roughly the inverse of their mean-energy ratio. Adopting a mean cosmic ray energy of $\sim 1$ GeV and an ambient temperature in the interstellar medium of $T \sim 10^4$ K, this ratio of number densities is $\sim 10^{-9}$. The ion gyro period in the interstellar medium is roughly $\sim 100$ s for a typical field of strength of $\sim 100$ pT. Cosmic rays streaming at a few times the Alfvén speed will create Alfvén waves in $\sim 10^{10}$ s, of order a few hundred years, long before they escape from the Galaxy.

The waves will then react back on the cosmic rays, scattering them in momentum space [Eq. (22.50)]. Now each time a particle is scattered by an Alfvén wave quantum, the ratio of its energy change to the magnitude of its momentum change must be the same as that in the waves and equal to the Alfvén speed, which is far smaller than the original energy to momentum ratio of the particle, $\sim c$ for a mildly relativistic proton. Therefore the effect of the Alfvén waves is to scatter the particle directions without changing their energies significantly. As the particles are already gyrating around the magnetic field, the effect of the waves is principally to change the angle between their momenta and the field (known as the *pitch angle*), so as to reduce their mean speed along the magnetic field.

Now when this mean speed is reduced to a value of order the Alfvén speed, the growth rate diminishes just like the growth rate of Langmuir waves is diminished after the electron

[3] Melrose 1984.

distribution function is flattened. Under a wide variety of conditions, cosmic rays are believed to maintain the requisite energy density in Alfvén wave turbulence to prevent them from streaming along the magnetic field with a mean speed much faster than the Alfvén speed (which varies between $\sim 3$ and $\sim 30$ km s$^{-1}$). This is a different model of their transport from spatial diffusion, which we assumed in Sec. 18.7, but the end result is similar and cosmic rays are confined to our galaxy for more than $\sim 10$ million years. These processes can be observed directly using spacecraft in the interplanetary medium.

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**EXERCISES**

**Exercise 22.7** *Problem: Stability of isotropic distribution function*
Consider an arbitrary isotropic distribution function and consider its stability to the growth of Langmuir waves. Show that the linear growth rates of all such waves are negative and so the plasma is stable to these modes.

**Exercise 22.8** *Challenge: Alfvén wave emission by streaming cosmic rays*
Consider a beam of high energy cosmic ray protons streaming along a background magnetostatic field in a collisionless plasma. Let the cosmic rays have an isotropic distribution function in a frame that moves with along the magnetic field with speed $u$, and assume that $u$ is large compared with the Alfvén speed but small compared with the speeds of the individual cosmic rays. Using the resonance condition (22.54) argue that there will be strong emission and absorption of Alfvén modes by the cosmic rays when their Larmor radii roughly match the wavelengths of the Alfvén waves.
Adapt the discussion of the emission of Langmuir waves by a bump on tail distribution to show that the growth rate is given to order of magnitude by Eq. (22.72).

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## 22.5 Parametric Instabilities

One of the approaches that is currently being pursued toward the goal of bringing about commercial nuclear fusion is to compress pellets of a mixture of deuterium and tritium using powerful lasers so that the gas densities and temperatures are large enough for the nuclear energy release to exceed the energy expended in bringing about the compression. At these densities the incident laser radiation behaves like a large amplitude plasma wave and is subject to a new type of instability that may already be familiar from dynamics, namely a *parametric instability.*

Consider how the incident light is absorbed by the relatively tenuous ionized plasma around the pellet. The critical density at which the incident wave frequency equals the plasma frequency is $\rho \sim 5\lambda_{\mu\mathrm{m}}^{-2}$ kg m$^{-3}$, where $\lambda_{\mu\mathrm{m}}$ is the wavelength measured in $\mu$m. For a wave incident energy flux $F \sim 10^{18}$ Wm$^{-2}$, the amplitude of the wave electric field will

**Box 22.1**
**Laser Fusion**

In the simplest scheme for Laser fusion, it is proposed that solid pellets of deuterium and tritium be compressed and heated to allow the reaction

$$d + t \rightarrow \alpha + n + 14MeV \tag{1}$$

to procede. An individual pellet would have mass $m \sim 5$ mg and initial radius $r_i \sim 2$ mm. If $\sim 1/3$ of the fuel burns then the energy released by a single pellet would be $\sim 500$ MJ.

As we described in Sec. 9.2, among the most powerful lasers available are the Q-switched, Neodymium-glass pulsed lasers which are capable of delivering an energy flux of $\sim 10^{18}$ W m$^{-2}$ at the pellet surface for several ns. These lasers operate at an infra red wavelength of $1.06\mu$m, however nonlinear optical methods (cf. Sec. 9.6) can be used to double or even triple the operating frequency. In a working reactor, ten lasers might be used, each one illuminating about 1 steradian with an initial radiation pressure of $P_{rad} \sim 3 \times 10^9$ N m$^{-2}$, which is less than a pellet's bulk modulus (cf. Table 10.1 in Sec. 10.4). Radiation pressure alone is therefore inadequate to compress the pellets.

However, at the high energy density involved, the incident radiation can be absorbed by plasma processes (see text), and the energy can be devoted to evaporating the surface layers of the pellet. The escaping hydrogen will carry away far more radial momentum which will cause the pellet to implode under the reaction force. This process is known as *ablation*. The maximum ablation pressure is $\sim 2P_{rad}c/v_e \sim 10^4 P_{rad}$ where $v_e \sim 30$ km s$^{-1}$ is the speed of the escaping gases.

Maximum compression will be achieved if the pellet remains cool. In this case, the dominant pressure will be the degeneracy pressure associated with the electrons. Compression factors of $\sim 10^4$ are contemplated, which are believed to be sufficient to initiate nuclear reactions.

be $E \sim (F/\epsilon_0 c)^{1/2} \sim 2 \times 10^{10}$ V m$^{-1}$. The velocity of a free electron oscillating in a wave this strong will be $v \sim eE/m_e\omega \sim 2000$ km s$^{-1}$ which is almost one per cent of the speed of light. It is therefore not surprising that nonlinear wave processes are important.

One of the most important such processes is called *stimulated Raman scattering*. In this process, the coherent electromagnetic wave with frequency $\omega$ convects a small pre-existing density fluctuation associated with a relatively low frequency Langmuir wave with frequency $\omega_{pe}$ and converts it into a current which varies at the beat frequency $\omega - \omega_{pe}$. This creates a new electromagnetic mode with this frequency. The vector sum of the $\mathbf{k}$ vectors of the two modes must also equal the incident $\mathbf{k}$ vector. When this can first happen, the new $\mathbf{k}$ is almost antiparallel to that of the incident mode and so the radiation is backscattered.

The new mode can combine nonlinearly with the original electromagnetic wave to produce a force $\propto \boldsymbol{\nabla} E^2$, which amplifies the original density fluctuation. Provided the growth rate of the wave is faster than the natural damping rates, e.g. that of Landau damping, there can be a strong back-scattering of the incident wave at a density well below the critical density of the incident radiation. (A further condition which must be satisfied is that the bandwidth of the incident wave must also be less than the growth rate. This will generally be true for a laser.) Stimulated Raman scattering is an example of a parametric instability. The incident wave frequency is called the *pump* frequency. One difference between parametric instabilities involving waves as opposed to just oscillations is that it is necessary to match spatial as well as temporal frequencies.

Reflection of the incident radiation by this mechanism reduces the ablation of the pellet and also creates a population of suprathermal electrons, which conduct heat into the interior of the pellet and inhibit compression. Various strategies, including increasing the wave frequency, have be devised to circumvent Raman back scattering (and also a related process called Brillouin back-scattering in which the Langmuir mode is replaced by an ion acoustic mode.

## 22.6 Solitons and Collisionless Shock Waves

In Sec. 20.3, we introduced ion-acoustic waves that have a phase speed $V_{\rm ph} \sim (k_B T_e/m_p)^{1/2}$, determined by a combination of the electron temperature $T_e$ and the proton mass $m_p$. In Sec. 21.3.6, we argued that these waves would be strongly Landau-damped unless the electron temperature greatly exceeded the proton temperature. However, this formalism was only valid for waves of small amplitude so that the linear approximation could be used. In Ex. 20.4, we considered the profile of a nonlinear wave and found a solution for a single ion-acoustic soliton valid when the waves are weakly nonlinear. We will now consider this problem in a slightly different way that is valid for strong nonlinearity in the wave amplitude. However we will restrict our attention to waves that propagate without change of form and so will not generalize the Korteweg-De Vries equation.

Once again, we use the fluid model and introduce an ion fluid velocity $u$. The electrons are supposed to be highly mobile and to assume a local density $\propto \exp(e\phi/k_B T_e)$, where $\phi$ is the electrostatic potential. The ions must satisfy equations of continuity and motion

$$\begin{aligned} \frac{\partial n}{\partial t} + \frac{\partial}{\partial z}(nu) &= 0 \; , \\ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} &= -\frac{e}{m_p}\frac{\partial \phi}{\partial z} \; . \end{aligned} \tag{22.73}$$

We now seek a solution for a wave moving with constant speed $V$, through the ambient plasma. In this case, all physical quantities must be functions of a single dependent variable $\xi = z - Vt$. If we allow a prime to denote differentiation with respect to $\xi$, then Eqs. (22.73) become

$$\begin{aligned} (u - V)n' &= -nu' \; , \\ (u - V)u' &= -\frac{e}{m_p}\phi' \; . \end{aligned} \tag{22.74}$$

These two equations can be integrated and combined to obtain an expression for the ion density $n$ in terms of the electrostatic potential

$$n = n_0(1 - 2e\phi/m_p V^2)^{-1/2} \; , \tag{22.75}$$

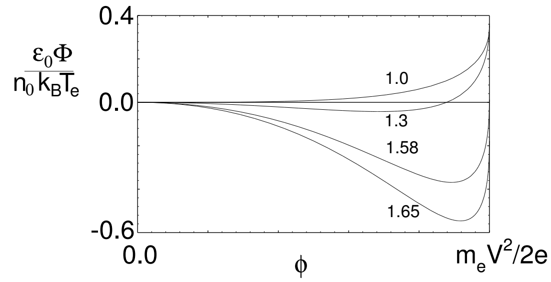


**Fig. 22.7:** Potential function $\Phi(\phi)$ used for exhibiting the properties of an ion-acoustic soliton for four different values of the ion acoustic Mach number $M$.

where $n_0$ is the ion density, presumed uniform, long before the wave arrives. The next step is to combine this ion density with the electron density and substitute into Poisson's equation to obtain a nonlinear ordinary differential equation for the potential:

$$\phi'' = -\frac{n_0 e}{\epsilon_0}\left\{\left(1-\frac{2e\phi}{m_p V^2}\right)^{-1/2} - e^{e\phi/k_B T_e}\right\} . \qquad (22.76)$$

Now the best way to think about this problem is to formulate the equivalent dynamical problem of a particle moving in a one dimensional potential well, where $\phi$ measures the position coordinate and $\xi$ is the time coordinate. As the right hand side of Eq. (22.76) varies only with $\phi$, we can treat it as minus the gradient of a scalar potential, $\Phi(\phi)$. Integrating the right hand side of Eq. (22.76) and assuming that $\Phi \to 0$ as $\phi \to 0$, (i.e. as $\xi \to \infty$), long before the arrival of the pulse, we obtain

$$\Phi(\phi) = \frac{n_0 k_B T_e}{\epsilon_0}\left[\left\{1-\left(1-\frac{2e\phi}{m_p V^2}\right)^{1/2}\right\}\frac{m_p V^2}{k_B T_e} - \left(e^{e\phi/k_B T_e} - 1\right)\right] . \qquad (22.77)$$

We have assumed that $0 < \phi < m_p V^2/2e$.

The shape of this potential well is sketched in Fig. 22.7 and is determined by the parameter $M = (m_p V^2/k_B T)^{1/2}$ which is immediately recognizable as the ion-acoustic Mach number, i.e. the ratio of the speed of the soliton to the ion-acoustic speed in the undisturbed medium. A solution for the potential profile $\phi(\xi)$ in the wave corresponds to the trajectory of a particle with zero total energy in this potential well. The particle starts at $\phi = 0$, with zero kinetic energy (i.e. $\phi' = 0$) and then accelerates to a maximum speed near the minimum in the potential before decelerating. If there is a turning point, the particle will come to rest, $\phi(\xi)$ will attain a maximum and then the particle will return to the origin. The particle trajectory corresponds to a symmetrical soliton, propagating with uniform speed.

Two conditions must be satisfied for a soliton solution. First, the potential well must be attractive. This will only happen when $d^2\Phi/d\phi^2(0) < 0$ which implies that $M > 1$. Second, there must be a turning point. This happens if $\Phi(m_p V^2/2e) > 0$. The maximum value of $M$ for which this is the case is a solution of the equation

$$e^{M^2/2} - 1 - M^2 = 0 \qquad (22.78)$$

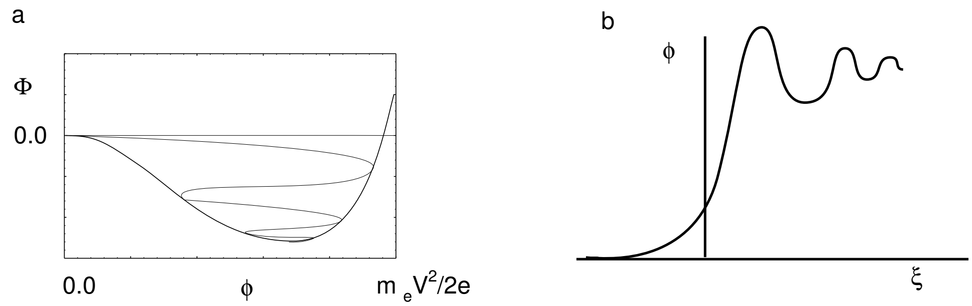


**Fig. 22.8:** Ion-acoustic shock waves. a) Solution in terms of equivalent potential. b) Electrostatic potential profile in shock.

or $M = 1.58$. Hence, ion acoustic soliton solutions only exist for

$$1 < M < 1.58. \tag{22.79}$$

The wave must travel sufficiently fast with respect to the ions that the particle can pass through the potential barrier. However the wave must not be so fast with respect to the electron thermal speed that the electrons are able to short out the potential near its maximum.

This analogy with particle dynamics is generally helpful. It also assists us in understanding a deep connection between solitons and laminar shock fronts. The equations that we have been solving so far contain the two key ingredients for a soliton, nonlinearity to steepen the wave profile and dispersion to spread it. However, they do not make provision for any form of dissipation, a necessary condition for a shock front where the entropy must increase. In a real collisionless plasma, this dissipation can take on many forms. It may be associated with anomalous resistivity or perhaps some viscosity associated with the ions. In many circumstances, some of the ions are reflected by the potential barrier and counterstream against the incoming ions which they eventually heat. Whatever its origin, the net effect of this dissipation will be to cause the equivalent particle to lose its total energy so that it can never return to its starting point. Given an attractive and bounded potential well, we find that the particle has no alternative except to sink toward to the bottom of the well. Depending upon the strength of the dissipation, the particle may undergo several oscillations before coming to rest.

The structure to which this type of solution corresponds is a laminar shock front. Unlike with a soliton, the wave profile in a shock wave is not symmetric in this case and instead describes a permanent change in the electrostatic potential $\phi$. Repeating the arguments above, we find that a shock wave can only exist when $M > 1$, that is to say, it must be supersonic with respect to the ion-acoustic sound speed. In addition there is a maximum *critical Mach number* close to $M = 1.6$ above which a laminar shock becomes impossible.

What happens when the critical Mach number is exceeded? Here there are several possibilities which include relying upon a more rapidly moving wave to form the shock front or appealing to turbulent conditions downstream from the shock front to enhance the dissipation rate.

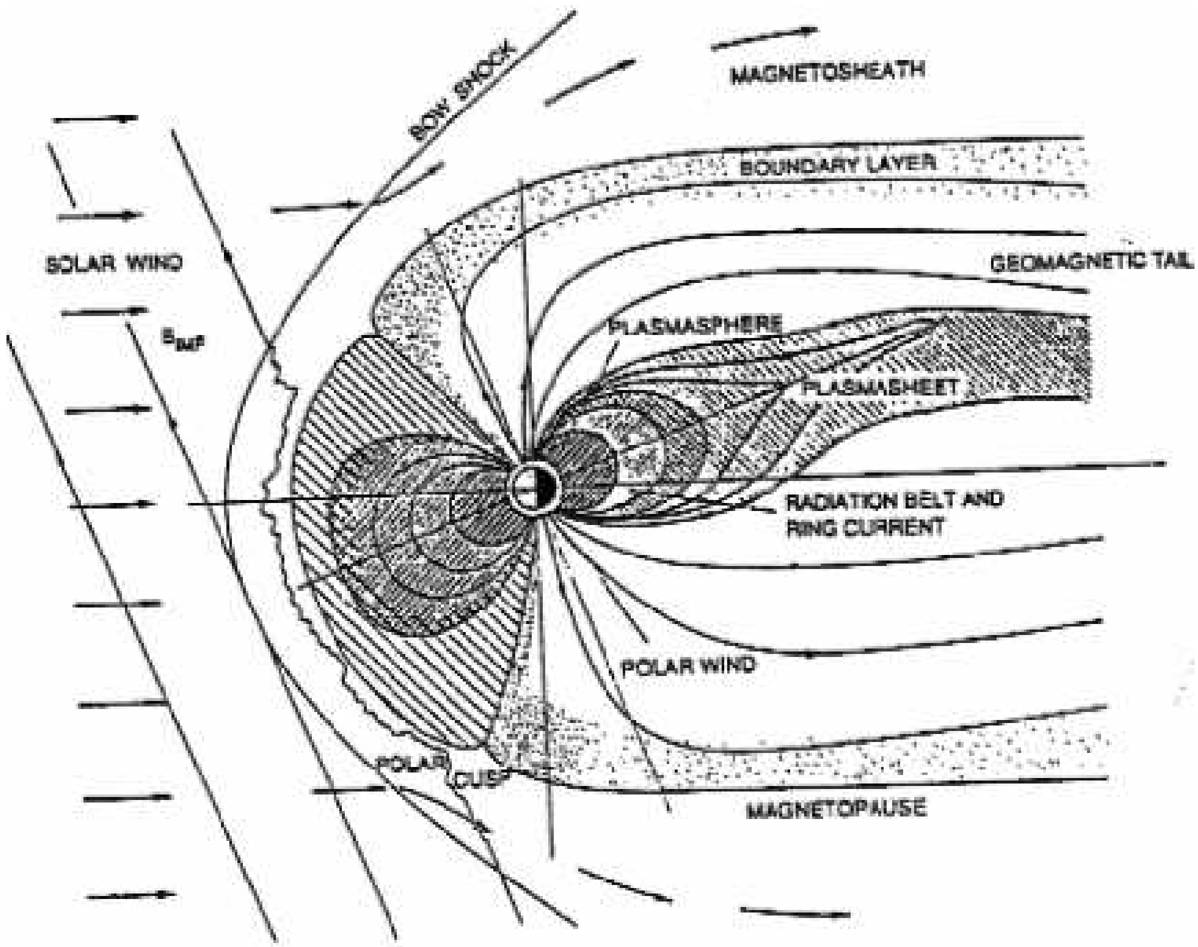


**Fig. 22.9:** Illustration of the form of the collisionless bow shock formed around the earth's magnetosphere. The earth's bow shock has been extensively studied using spacecraft. Alfvén, ion-acoustic, whistler and Langmuir waves are all generated with large amplitude in the vicinity of shock fronts by the highly non-thermal particle distributions. (Adapted from Parks 1991.)

This ion-acoustic shock front is the simplest example of a collisionless shock. Essentially every wave mode can be responsible for the formation of a shock. The dissipation in these shocks is still not very well understood but we can observe them in several different environments within the heliosphere and also in the laboratory. The best studied of these shock waves are those based on hydromagnetic waves which were introduced briefly in chapter 10. The solar wind moves with a speed that is typically 5 times the Alfvén speed. It should therefore form a bow shock (one based upon the fast magnetosonic mode), whenever it encounters a planetary magnetosphere. This happens despite the fact that the mean free path of the ions in the solar wind is typically much larger than the size of the shock front. The thickness of the shock front turns out to be a few ion Larmor radii. This is a dramatic illustration of the importance of collective effects in controlling the behavior of essentially collisionless plasmas; see Fig. 22.9.

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**EXERCISES**

**Exercise 22.9** *Derivation: Critical Mach number for an Ion-acoustic shock wave*
Verify Eq. (22.77) and show numerically that the critical Mach number for a laminar shock front is $M = 1.56$.

**Exercise 22.10** *Problem: Solar-wind termination shock*
The solar wind is a quasi-spherical outflow of plasma from the sun. At the radius of the earth's orbit, the mean proton and electron densities are $n_p \sim n_e \sim 4 \times 10^6$ m$^{-3}$, their temperatures are $T_p \sim T_e \sim 10^5$ K, and their common radial fluid speed is $\sim 400$ km s$^{-1}$. The mean magnetic field strength is $\sim 1$ nT. Eventually, the radial momentum flux in the solar wind falls to the value of the mean *interstellar* pressure, $\sim 10^{-13}$ N m$^{-2}$ and a shock will develop.

(a) Estimate the radius where this will occur.

(b) The solar system moves through the interstellar medium with a speed $\sim 30$ km s$^{-1}$. Sketch the likely flow pattern near this radius.

(c) How do you expect the magnetic field to vary with radius in the outflowing solar wind? Estimate its value at the termination shock.

(d) Estimate the electron plasma frequency, the ion acoustic Mach number and the proton Larmor radius just ahead of the termination shock front and comment upon the implications of these values for the shock structure.

(e) The Voyager 1 spacecraft was launched in 1977 and is now moving radially away from the sun with a speed $\sim 15$ km s$^{-1}$. When do you think it will pass through the termination shock?

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