Physics 136a: Notes on the Maxwell-Bloch Equations for a Laser

The Maxwell-Bloch equations describe the dynamics of a simple two-level laser. To derive the equations we need to do a little quantum mechanics, but the resulting equations are classical equations describing the macroscopic (classical) electric field amplitude, polarization, and population inversion.

We start with the quantum mechanical Hamiltonian for the two levels in a single atom. We take the energies of the levels as $\pm\Delta/2$ and suppose they are connected by a dipole matrix element M, which we arrange to be real. Expressed in matrix notation with the basis the two levels the Hamiltonian is

$$H = \begin{bmatrix} \frac{\Delta}{2} & -ME \\ -ME & -\frac{\Delta}{2} \end{bmatrix} \tag{1}$$

with E the electric field. A two level system is analogous to a spin-$\frac{1}{2}$ system, and so we can express this Hamiltonian in terms of *fictitious* spin-$\frac{1}{2}$ operators

$$s_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad s_+ = s_x + is_y = \hbar\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad s_- = s_x - is_y = \hbar\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{2}$$

as

$$H = \frac{\Delta}{\hbar}s_z - \frac{ME}{\hbar}(s_+ + s_-). \tag{3}$$

The time dependence of the system is given in the Heisenberg picture by the equations of motion of the operators

$$i\hbar\dot{\mathbf{s}} = [\mathbf{s}, H]. \tag{4}$$

We therefore need the commutation expressions for the spin operators which are easy to work out from the matrix expressions (2)

$$[s_z, s_\pm] = \pm\hbar s_\pm, \tag{5}$$

$$[s_+, s_-] = 2\hbar s_z. \tag{6}$$

The equations of motion are then

$$i\hbar\dot{s}_z = -ME(s_+ - s_-), \tag{7}$$

$$i\hbar\dot{s}_+ = -\Delta s_+ - 2MEs_z, \tag{8}$$

$$i\hbar\dot{s}_- = \Delta s_- + 2MEs_z. \tag{9}$$

We now sum over all the spins in a unit volume assuming that the electric field is coherent over this region so that the spins are interacting with the *same* electric field. The large total spin can be taken as a classical variable, i.e., we neglect the $O(\hbar)$ quantum fluctuations compared with the mean. Writing the sum $\sum \mathbf{s} = \mathbf{S}$ gives us the classical equations of motion

$$\dot{S}_z = i\frac{ME}{\hbar}(S_+ - S_-), \tag{10}$$

$$\dot{S}_+ = i\frac{\Delta}{\hbar}S_+ + 2i\frac{ME}{\hbar}S_z, \tag{11}$$

$$\dot{S}_- = -i\frac{\Delta}{\hbar}S_- - 2i\frac{ME}{\hbar}S_z. \tag{12}$$

(Note that although the coefficients contain $\hbar$ they are just the frequencies corresponding to the quantum energies, and so have a sensible classical interpretation). We now suppose an electric field given by a slowly varying amplitude of a cavity mode at frequency ω with the lasing frequency $\Delta/\hbar = \omega + \Omega$ with Ω small (small detuning). We also introduce slowly varying spin amplitudes $\bar{S}_\pm$

$$E = \frac{1}{2}(A(t)e^{-i\omega t} + A^{*(t)}e^{i\omega t}), \qquad S_\pm = \bar{S}_\pm(t)e^{\pm i\omega t}. \tag{13}$$

Then keeping only resonant terms in the equations (i.e., dropping the terms varying as $e^{\pm 2i\omega t}$) gives

$$\dot{S}_z = \frac{1}{2}\frac{M}{\hbar}i(A\bar{S}_+ - A^*\bar{S}_-), \tag{14}$$

$$\dot{\bar{S}}_+ = i\Omega\bar{S}_+ + i\frac{M}{\hbar}A^*S_z, \tag{15}$$

$$\dot{\bar{S}}_- = -\Omega\bar{S}_- - i\frac{M}{\hbar}AS_z. \tag{16}$$

This approximation is sometimes called the *rotating wave* approximation.

We now return to the native variables for the laser system. The population inversion is $N = (2/\hbar)S_z$ and the polarization (cf., the term coupling to the electric field in Eq. (3) is $(M/\hbar)(S_+ + S_-)$ which can be written in a form analogous to the electric field as $\frac{1}{2}(Pe^{-i\omega t} + P^*e^{i\omega t})$ with $P = (2M/\hbar)\bar{S}_-$. Thus we have

$$\dot{P} = -i\Omega P - \frac{iM^2}{\hbar}AN. \tag{17}$$

$$\dot{N} = \frac{i}{2\hbar}(AP^* - A^*P), \tag{18}$$

In addition we have Maxwell's equation for the electric field, Eq. (9.19). If there is no spatial variation of the wave amplitude A this reduces to (cf., Eq. (9.43a) or the derivation of the nonlinear Schrodinger equation)

$$\dot{A} = \frac{i\omega}{2\varepsilon_0}P. \tag{19}$$

The equations derived so far are lossless—there are no sources of dissipation. To get the full behavior we supplement the equations by phenomenological damping terms. This gives the *Maxwell-Bloch equations*

$$\dot{A} + \gamma A = \frac{i\omega}{2\varepsilon_0}P, \tag{20}$$

$$\dot{P} + \gamma_\perp P = -i\Omega P - \frac{iM^2}{\hbar}AN, \tag{21}$$

$$\dot{N} + \gamma_\parallel(N - N_0) = \frac{i}{2\hbar}(AP^* - A^*P), \tag{22}$$

with γ_i positive damping constants. Note that in the population inversion equation the dissipation tends to return N to N_0, a value determined by the pumping. The damping γ of the electric field is due to absorption at the cavity mirrors and the light exciting the cavity for our use. The damping constants $\gamma_{\perp,\parallel}$ may be due to atomic collisions, spontaneous emission and other processes. For the case of no detuning $\Omega = 0$ we may take $\bar{P} = iP$ and A to be real, and the equations reduce to three real equations

$$\dot{A} + \gamma A = \frac{\omega}{2\varepsilon_0}\bar{P}, \tag{23}$$

$$\dot{\bar{P}} + \gamma_\perp\bar{P} = \frac{M^2}{\hbar}AN, \tag{24}$$

$$\dot{N} + \gamma_\parallel(N - N_0) = \frac{1}{\hbar}A\bar{P}. \tag{25}$$

Either set of equations are *coupled nonlinear* ordinary differential equations, for which explicit solutions cannot be found in general. They can be simplified in certain special cases.

Class B Lasers

In this case (eg. relevant to semiconducting lasers) $\gamma_\perp \gg \gamma, \gamma_\parallel$. The variables will evolve on the slow time scale set by $\gamma, \gamma_\parallel$ and then in the equation for $\bar{P}$ we can neglect the time derivative term compared with the $\gamma_\perp \bar{P}$ term. We can now solve explicitly for $\bar{P}$ and substitute into the other two equations (This approach is known as *adiabatic elimination*.). For the no-detuning case this gives

$$\dot{A} + \gamma A = \frac{M^2}{2\varepsilon_0 \hbar} \frac{\omega}{\gamma_\perp} N A, \tag{26}$$

$$\dot{N} + \gamma_\parallel (N - N_0) = -\frac{M^2}{\hbar^2} \frac{1}{\gamma_\perp} N A^2. \tag{27}$$

The first equation shows the *amplification* of A and the second equation the depopulation of the upper level by feeding energy into the electromagnetic field. The onset of lasing (put $N = N_0$ in the first equation) occurs for $N_0 > N_c$ with

$$\frac{M^2 N_c}{2\varepsilon_0 \hbar} \omega > \gamma \gamma_\perp. \tag{28}$$

You can check that the first fraction in this expression is indeed a frequency as required for dimensional consistency.

Class A Lasers

For class A lasers (eg., dye lasers) both atomic relaxation rates are fast compared with the cavity relaxation $\gamma_\perp, \gamma_\parallel \gg \gamma$. In this case we can ignore the $\dot{N}$ term as well, and solve explicitly for N (again we look at the no-detuning case)

$$N = \frac{N_0}{1 + \frac{M^2}{\hbar^2} \frac{1}{\gamma_\parallel \gamma_\perp} N A^2}. \tag{29}$$

Substituting in the A equation gives

$$\dot{A} + \gamma A = \frac{M^2}{2\varepsilon_0 \hbar} \frac{\omega}{\gamma_\perp} \frac{N_0}{1 + \frac{M^2}{\hbar^2} \frac{1}{\gamma_\parallel \gamma_\perp} N A^2} A. \tag{30}$$

This gives the same lasing threshold, but now we see the amplification of A is reduced as A^2 increases leading to the saturation of A to a steady state value. For N_0 near N_c the value of A^2 will be small, and the RHS of Eq. (30) can be expanded to give

$$\gamma^{-1} \dot{A} = \left(\frac{N_0}{N_c} - 1 \right) A - \frac{M^2}{\hbar^2} \frac{1}{\gamma_\parallel \gamma_\perp} A^3. \tag{31}$$

This equation explicitly shows the growth of the amplitude for $N_0 > N_c$ and then saturation (set $\dot{A} = 0$) at the value $A_{\text{sat}} \propto (N_0 - N_c)^{1/2}$. This type of sharp onset is reminiscent of a second order phase transition.

Class C Lasers

For class C lasers all the relaxation rates are comparable, and all three dynamical equations must be used. Steady state lasing solutions can be found, but if $\gamma > \gamma_\parallel + \gamma_\perp$ these solutions become *unstable* for large enough N_0 to *time periodic* or even *chaotic* solutions. Equations (23-25) in fact are identical to the famous equations used by Lorenz in pioneering work that discovered chaos in a fluid dynamics system (see Assignment 9). Fortunately for the application of lasers this “bad cavity” limit does not usually apply to realistic lasers, although they can be constructed in this limit to investigate the phenomenon of laser chaos.