Solution for Chapter 10

(compiled by Xinkai Wu, modified by Jeff Atwell for 2004-2005 course, modified by Huan Yang for 2006-2007 course)

A.

10.19. Dimensionally Reduced Shape Equation for a stressed Plate. [by Huan Yang]

a. In every vertical plane through the plate, e.g. the x, z plane, the geometric argument depicted in Eq. (10.72) and Fig. 10.9 of the text remains correct for the plate, as for the cantilever. This argument implies, for example, that $\xi_{x,x} = z/\mathcal{R}_x = zd^2\eta/dx^2$, and $\xi_{y,y} = z/\mathcal{R}_y = zd^2\eta/dy^2$, etc.; where $\mathcal{R}_x$ and $\mathcal{R}_y$ are the radius of curvature of the plate's intersection with the vertical x, z plane and y, z plane. Consider a slice in a plane spanned by the z direction and some horizontal direction $\mathbf{n}$ (a unit vector). The claimed general answer $\xi_{a,b} = z\eta_{a,b}$ (note the sign error in the exercise) implies that $\xi_{a,b}n_a n_b = z\eta_{a,b}n_a n_b$, which is equivalent to $\xi_{s,s} = z\eta_{s,s}$ for s being a Cartesian coordinate along the n direction — the desired result for any vertical plane. Thus, the claimed general answer must be correct.

b. Note T_{zz} is 0 both on the upper and lower surface. Since h is small, a finite T_{zz} in the material would generate a large $T_{zz,z}$ which would affect the f balance in z direction. So T_{zz} must be negligible.

$$T_{zz} = -K\theta - 2\mu\Sigma_{zz} = \frac{-E\theta}{3(1-2\nu)} + \frac{E}{(1+\nu)}(\xi_{a,a} - \frac{2}{3}\Theta) = 0 \ , \tag{1}$$

Note $\xi_{a,a} = -z\eta_{,aa} = -z\nabla^2\eta$, we can get:

$$\Theta = -\left(\frac{1-2\nu}{1-\nu}\right) z\nabla^2\eta \ , \tag{2}$$

Note $T_{ab} = -K\Theta\delta_{ab} - 2\mu\Sigma_{ab}$,

$$\mu = \frac{E}{2(1+\nu)}, K = \frac{E}{1-2\nu}\Theta = -\left(\frac{1-2\nu}{1-\nu}\right) z\nabla^2\eta \ , \Sigma_{ab} = \xi_{a,b} - \frac{1}{3}\Theta\delta_{ab} \tag{3}$$

we can easily get:

$$T_{ab} = Ez\left[\frac{\nu}{(1-\nu^2)}\nabla^2\eta\ \delta_{ab} + \frac{\eta_{,ab}}{(1+\nu)}\right] \ . \tag{4}$$

c.

$$f_a = -\nabla_a T = -T_{ab,b} - T_{az,z} = -\frac{Ez}{1-\nu^2}\nabla^2\eta_{,a} - T_{az,z} = 0, \tag{5}$$

$$\frac{Ez^2}{1-\nu^2}\nabla^2\eta_{,a} = -zT_{az,z} \tag{6}$$

Integrate over z from $-\frac{h}{2}$to$\frac{h}{2}$, Integrate by parts for the right hand side, we can get:

$$\int T_{za}dz = \frac{Eh^3}{12(1-\nu^2)\nabla^2\eta_{,a}} \tag{7}$$

So $D = \frac{Eh^3}{12(1-\nu^2)}$.

d.

$$M_{ab} = \int zT_{ab}dz = \int Ez^2(\frac{\nu}{1-\nu^2}\nabla^2\eta\delta_{ab}+\frac{\eta_{,ab}}{1+\nu})dz = \frac{Eh^3}{12}(\frac{\nu}{1-\nu^2}\nabla^2\eta\delta_{ab}+\frac{\eta_{,ab}}{1+\nu}), \tag{8}$$

$$M_{ab,b} = \frac{Eh^3}{12}(\frac{\nu}{1-\nu^2}+\frac{1}{1+\nu})\nabla^2\eta_{,a} = D\nabla^2\eta_{,a} = S_a. \tag{9}$$

e.

$$\Delta F = \oint S_a dl = \int \nabla_a S_a dS, \frac{\Delta F}{\Delta S} = F = S_{a,a} = D\nabla^2\eta_{,aa} = D\nabla^2\nabla^2\eta. \tag{10}$$

f. From force balance in z direction:

$$(\int_w^{\frac{h}{2}} T_{za}dz)_{,a} + T_{zz}(z=w) = F \tag{11}$$

so $T_{zz} \sim F$. Also We know η typically changes in a lengthscale R, so $\nabla^2\eta \sim \frac{\eta}{R^2}$.

$$T_{ab} \sim \frac{D}{h^2}\nabla^2\eta \sim \frac{D}{h^2}\frac{\eta}{R^2}; F \sim D\nabla^2\nabla^2\eta \sim \frac{D}{R^4}\eta \sim O(\frac{h^2}{R^2}T_{ab}). \tag{12}$$

So T_{zz} is very small.

B.

10.17 Elastica [by Xinkai Wu/02]

(i) Consider the part of the wire between one end and the point a distance z' from this end. The total force (the force applied at the end and the stress force applied by the rest of the wire) exerted on this part must vanish. The $\mathbf{e}_z$ component of this immediately gives

$$F cos\theta = \int T_{z'z'}dx'dy'$$

while the $\mathbf{e}_x$ component gives

$$F sin\theta = \int T_{x'z'}dx'dy'$$

Now consider the infinitesimal segment between z' and $z'+dz'$. The total torque exerted on this segment is

$$Fsin\theta dz' + M(z') - M(z+dz')$$

where $M = \int x' T_{z'z'} dx'dy'$ (and by using $T_{z'z'} = -E\xi_{z',z'} = -Ex'\frac{d\theta}{dz'}$ and performing the integral, one gets $M = -D\frac{d\theta}{dz'}$ with the flexural rigidity $D = \frac{Eba^3}{12}$).

This total torque must vanish, which gives

$$Fsin\theta = \frac{dM}{dz'}$$

Combining the above results immediately gives

$$\frac{d^2\theta}{dz'^2} = -\frac{Fsin\theta}{D} = -\frac{sin\theta}{l^2}$$

where we have defined the characteristic length $l \equiv \sqrt{\frac{D}{F}}$. For rubber, $E = 0.002GPa$, and if we take $a = 1cm$, $b = 0.5cm$, and apply a force $F = 10N$, we get $l \approx 9mm$.

(ii) Mathematica gives a solution

$$\theta(z') = 2 \cdot am(\frac{1}{\sqrt{2}}\frac{z'}{l}|2)$$

where $am(u|m)$ is the inverse function of the elliptic integral of the first kind $F(\phi|m)$, namely $\phi = am(u|m) \Leftrightarrow u = F(\phi|m)$. (Recall that $F(\phi|m) = \int_0^\phi \frac{1}{\sqrt{1-msin^2t}}dt$).

(iii) Now to get the shape of the wire we need to find $z(x)$.

In the previous part we obtained $\theta(z')$, whose inverse is just

$$z' = \sqrt{2}l\int_0^{\theta/2}\frac{1}{\sqrt{1-2sin^2t}}dt = \sqrt{2}l\int_0^{\theta/2}\frac{1}{\sqrt{cos2t}}dt$$

(note that this solution has $\theta(z'=0)=0$.) so we get

$$dz' = \frac{l}{\sqrt{2}}\frac{1}{\sqrt{cos\theta}}d\theta$$

therefore we have

$$\begin{aligned} cos\theta &= \frac{dx}{dz'} = \sqrt{2cos\theta}\frac{1}{l}\frac{dx}{d\theta} \\ sin\theta &= \frac{dz}{dz'} = \sqrt{2cos\theta}\frac{1}{l}\frac{dz}{d\theta} \end{aligned}$$

Integrating the second equation using the initial condition $\theta(z=0)=0$ gives

$$cos\theta = \left(\frac{z}{\sqrt{2}} - 1\right)^2$$

Using this result we get

$$\frac{dx}{dz} = \frac{dx/d\theta}{dz/d\theta} = \frac{cos\theta}{sin\theta} = \frac{\left(\frac{z}{\sqrt{2}} - 1\right)^2}{\pm\sqrt{1 - \left(\frac{z}{\sqrt{2}} - 1\right)^4}}$$

Integrating the above equation numerically using Mathematica, we get the plot for $x(z),\ z \in [0, 2\sqrt{2}l]$(which corresponds to positive θ), see Fig. 1

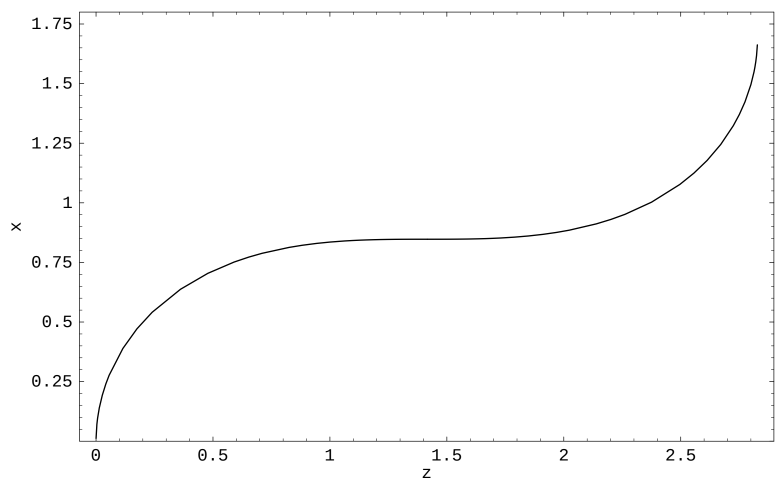

Figure 1: Elastica $x(z)$ in units of l; $z \in [0, 2\sqrt{2}l]$

As x continues increasing, z will decrease(which corresponds to negative θ), and we see that $z(x)$ is a periodic function as that in (b) of Fig. 10.7 of the text, with period $\approx 2 \times 1.7l$, and the crest height is $z_{max} = 2\sqrt{2}l$.

(iv) If anyone of you have a slender wire good for this experiment and would like to give me a demonstration, I'd be very glad to see it!

10.20 Paraboloidal mirror [[by Alexei Dvoretskii/00]

(i) The equation for a paraboloid with focal length f is

$$z = \frac{r^2}{4f}$$

and for a sphere of radius R,

$$z = R - \sqrt{R^2 - r^2}$$

Choosing $R = 2f$ and expanding for $\frac{r}{R} << 1$, we get

$$z = \frac{r^2}{4f} + \frac{r^4}{64f^3}$$

The vertical displacement of the mirror is therefore $\eta(r) = \frac{r^4}{64f^3}$

(ii) Because of the cylindrical symmetry the laplacian has the simple form

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}$$

The pressure that must be applied is then given by

$$F = D\nabla^2\nabla^2\eta = \frac{D}{f^3}$$

(ii) The total force is

$$F\pi R^2 = \pi R^2 \frac{D}{f^3} = NS_{zr}$$

Therefore the force applied at each lever is

$$S_{zr} = \frac{\pi D R^2}{N f^3}$$

The associated bending torque is just

$$M = S_{zr} R = \frac{\pi D R^3}{N f^3}$$

(iv) The radial displacement is found from

$$\xi_r = -z\frac{\partial \eta}{\partial r} = -\frac{r^3 z}{16 f^3}$$

(v) The associated expansion and strain are

$$\begin{aligned}
\Theta &= \frac{1}{r}\frac{\partial}{\partial r} r\xi_r = \frac{-r^2 z}{4f^3} \\
\Sigma_{rr} &= \frac{2\partial \xi_r}{3\partial r} - \frac{\xi_r}{3r} = -\frac{5r^2 z}{48 f^3} \\
\Sigma_{\phi\phi} &= \frac{2\xi_r}{3r} - \frac{\partial \xi_r}{3\partial r} = -\frac{r^2 z}{48 f^3} \\
\Sigma_{zz} &= -\frac{\partial \xi_r}{3\partial r} - \frac{\xi_r}{3r} = \frac{r^2 z}{12 f^3}
\end{aligned}$$

Using

$$\mathbf{T} = -K\Theta\mathbf{g} - 2\mu\mathbf{\Sigma}$$

we see that the maximum stress is T_{rr} at the rim

$$T_{max} = \frac{R^2 h}{8 f^3}\left(K + \frac{5}{6}\mu\right)$$

After straightforward manipulation

$$T_{max} = \frac{3-2\nu}{256(1-2\nu)(1+\nu)} E \left(\frac{2R}{f}\right)^3 \frac{h}{R} = \frac{(3-2\nu)R^2 hE}{32(1-2\nu)(1+\nu)f^3}$$

Now, if the mirror is not to break then it should be that $T_{max} \leq 10^{-4}E$. $\nu = 0.25$ for glass. Then there's a limit on how "fast" and thick a mirror can be at the same time

$$\left(\frac{2R}{f}\right)^3 \frac{h}{R} \leq 1.5 \times 10^{-2}$$

C.

Problem 11.2: Influence of gravity on wave speed[by Guodong Wang/03]
Including the effect of gravity into equation (BT-11.4),

$$\rho\frac{\partial^2 \xi}{\partial t^2} = (K + \frac{1}{3}\mu)\nabla(\nabla \cdot \xi) + \mu\nabla^2\xi + \rho\mathbf{g} \quad (13)$$

Consider small pertubations to the displacement and density, Substituing $\rho \to \rho_0 + \delta\rho$, $\xi \to \xi_0 + \xi$ into Eq. (13), Keeping the small pertubations to the first order, we obtain the static equation

$$0 = (K + \frac{1}{3}\mu)\nabla(\nabla \cdot \xi_0) + \mu\nabla^2\xi_0 + \rho_0\mathbf{g} \quad (14)$$

which describe the static displacement ξ_0 produced by gravity $\mathbf{g}$, and the wave equation

$$\rho_0\frac{\partial^2 \xi}{\partial t^2} = (K + \frac{1}{3}\mu)\nabla(\nabla \cdot \xi) + \mu\nabla^2\xi + \delta\rho\mathbf{g} \quad (15)$$

Recalling Eq. (BT-11.3), $\delta\rho \simeq -\rho_0\nabla \cdot \xi$, Eq.(15) can be written as

$$\rho_0\frac{\partial^2 \xi}{\partial t^2} = (K + \frac{1}{3}\mu)\nabla(\nabla \cdot \xi) + \mu\nabla^2\xi - \rho_0\nabla \cdot \xi\mathbf{g}. \quad (16)$$

The first term and the second term on the right side of Eq.(16) are of the same order of magnitude. The ratio of the third to the second term is of the order of $\frac{\rho_0 g}{\mu k} \sim \frac{\lambda}{\mu/\rho_0 g}$. For short wavelength modes, $\lambda \ll \mu/\rho_0 g$, the third term can be ignored and equation (16) reduces to Eq. (BT-11.4).

Taking $\mu \sim 100GPa$ (see BT table 10.1), $\rho_0 \sim 3g/cm^3$, $\mu/\rho_0 g \sim 1000km$. So only for $\lambda \gtrsim 1000km$, the gravitational terms are important.

0.1 Problem 11.7: Xylophone

[by Alexander Putilin/01, modified by Geoffrey Lovelace]

We start with the wave equation (BT-11.31)

$$\frac{\partial^4 \eta}{\partial z^4} + \frac{\Lambda}{D}\frac{\partial^2 \eta}{\partial t^2} = 0 \tag{17}$$

We are looking for the standing-wave solution of the form

$$\eta(t, z) = e^{-i\omega t} f(z) \tag{18}$$

Substituting into (17) we get the O.D.E. for $f(z)$

$$\frac{d^4 f}{dz^4} = \frac{\Lambda \omega^2}{D} f \tag{19}$$

It has the general solution

$$f(z) = A \sin kz + B \cos kz + C \sinh kz + D \cosh kz \tag{20}$$

where A, B, C, D are constants and

$$\omega^2 = \frac{D}{\Lambda} k^4 \tag{21}$$

To define the spectrum of k we should impose boundary conditions. We have two different cases.

0.2 Ends are free.

The appropriate boundary conditions are:

$$f(0) = f(l) = 0 \qquad \text{- ends are fixed} \tag{22}$$

$$f''(0) = f''(l) = 0 \qquad \text{longitudinal stress vanishes} \tag{23}$$

(because $T_{zz} = -Ex\frac{d^2\eta}{dz^2}$ by Eq. (BT-10.71) with interchange of x and z)

$$\Longrightarrow B = C = D = 0, \qquad kl = \pi n, \qquad n = 1, 2, 3, \ldots \tag{24}$$

So the frequency spectrum is

$$\omega_n = \sqrt{\frac{D}{\Lambda}} k^2 = \sqrt{\frac{D}{\rho A}} \left(\frac{\pi n}{l}\right)^2 \tag{25}$$

0.3 Clamped ends.

In this case the boundary conditions are:

$$f(0) = f(l) = 0 \tag{26}$$

$$f'(0) = f'(l) = 0 \tag{27}$$

They can be nontrivially satisfied only if

$$\cosh(kl)\cos(kl) = 1 \tag{28}$$

so the spectrum is

$$\omega_n = \sqrt{\frac{D}{\rho A}}\left(\frac{\alpha_n}{l}\right)^2, \tag{29}$$

where α_n are solution of the eqn.

$$\cosh(\alpha)\cos(\alpha) = 1 \tag{30}$$

If $\alpha \gg 1$, then $\cosh\alpha \gg 1$ and $\cos\alpha \ll 1$ i.e. $\alpha_n \approx \pi(n+\frac{1}{2})$. Actually this formula gives good approximation for all $n = 1, 2, 3, \ldots$.

$$\Longrightarrow \omega_n \approx \sqrt{\frac{D}{\rho A}}\left[\frac{\pi}{l}(n+\frac{1}{2})\right]^2, \tag{31}$$

The ratios of eigen-frequencies are

$$\omega_1 : \omega_2 : \omega_3 : \cdots \approx 9 : 25 : 49 : \ldots$$

The higher harmonics become progressively weaker in volume. When the rod is unclamped, the next loudest harmonic has a frequency four times the primary frequency; this is two octaves higher. In the clamped case, the higher frequencies are not integer multiples of the primary frequency, which makes the sound much less pleasant.

D.

0.4 Problem 11.3. Solving the algebraic wave equation by Matrix Techniques.

[by Geoffrey Lovelace]

a. The algebraic wave equation is given in Eq. (11.6). In index notation, this equation is

$$\rho\omega^2\xi_i = \mu k^2\xi_i + \left(K + \frac{1}{3}\mu\right)k_i k_j \xi_j. \tag{32}$$

Here, ξ_i is the displacement vector, ω is the frequency of the mode under consideration, k_i is the corresponding wave vector (with magnitude k), K is the bulk modulus, and μ is the shear modulus.

To express this in the form $M_{ij}\xi_j = 0$, introduce some convenient Kronecker deltas to find

$$\left[\left(\mu k^2 - \rho\omega^2\right)\delta_{ij} + \left(K + \frac{1}{3}\mu\right)k_i k_j\right]\xi_j = 0$$

$$\Rightarrow M_{ij} = \left(\mu k^2 - \rho\omega^2\right)\delta_{ij} + \left(K + \frac{1}{3}\mu\right)k_i k_j. \tag{33}$$

b. This matrix equation has a nontrivial solution if and only if the determinant $M \equiv \det(M_{ij})$ vanishes. This follows from the fact that M is invertible if and only if its determinant vanishes (see any introductory linear algebra text for a proof), and if M is invertible, then the only solution is the trivial solution: if M^{-1} exists, you can write the matrix equation as $M_{ki}^{-1}M_{ij}\xi_j = \delta_{kj}\xi_j = \xi_k = 0$.

The determinant of M_{ij} is easy to work out, since M_{ij} is just a 3×3 matrix. The result is

$$M = 0 \Rightarrow \frac{1}{3}(3Kk^2 + 4\mu k^2 - 3\rho\omega^2)(\mu k^2 - \rho\omega^2)^2 = 0. \tag{34}$$

This is clearly a cubic equation for ω^2. One root is

$$\omega_L = \sqrt{\frac{(K + \frac{4}{3}\mu)}{\rho}}k = c_L k, \tag{35}$$

and the other two roots are

$$\omega_T = \sqrt{\frac{\mu}{\rho}}k = c_T k. \tag{36}$$

The "L" and "T" labels anticipate the fact that these eigenfrequencies correspond, respectively, to longitudinal and transverse waves.

c. Let's let $\mathbf{k} = k\hat{\mathbf{z}}$. Then, in matrix notation,

$$[M_{ij}] = \text{diag}\left[k^2\mu - \rho\omega^2, k^2\mu - \rho\omega^2, k^2\left(K + \frac{4}{3}\mu\right) - \rho\omega^2\right]. \tag{37}$$

Inserting $\omega = c_L k$ yields

$$[M_{ij}]_L = \text{diag}\left[-k^2\left(K + \frac{1}{3}\mu\right), -k^2\left(K + \frac{1}{3}\mu\right), 0\right]. \tag{38}$$

To satisfy $[M_{ij}]_L\xi_j = 0$, ξ has the form $\xi = \xi\hat{\mathbf{z}}$. In other words, ξ is a longitudinal displacement.

d. Following identical reasoning as in part c, $\omega = c_T k$ yields

$$[M_{ij}]_T = \text{diag}\left[0, 0, \left(K + \frac{1}{3}\mu\right)\right]. \tag{39}$$

To satisfy $[M_{ij}]_T\xi_j = 0$, ξ has the form $\xi = \xi_x\hat{\mathbf{x}} + \xi_y\hat{\mathbf{y}}$. In other words, ξ is a transverse displacement.

0.5 Problem 11.6: Speeds of elastic waves

[by Alexander Putilin/01]

We need to find sound velocities of 5 types of elastic waves. Consider them in turn.

0.5.1 Longitudinal waves along a rod.

The correspoding formula has been derived in BT. (BT-11.22)

$$c_1 = \sqrt{\frac{E}{\rho}} \tag{40}$$

0.5.2 Longitudinal waves along a sheet

Choose coordinate system so that z-axis is orthogonal to the sheet, x- and y-axes are parallel and wave is propagating in x-direction. Since wave is longitudinal it means

$$\xi_y = 0 \tag{41}$$

No force acts on the face of the sheet, i.e. $T_{zi} = 0$.

$$\Longrightarrow T_{zz} = -K\theta - 2\mu\Sigma_{zz} = 0, \qquad \Sigma_{zz} = -\frac{K}{2\mu}\theta \tag{42}$$

Tracelessness of Σ implies

$$\Sigma_{xx} + \Sigma_{yy} + \Sigma_{zz} = 0 \tag{43}$$

But $\Sigma_{yy} = \xi_{y,y} - \frac{1}{3}\theta = -\frac{1}{3}\theta$. (Cf.(41))
Plugging into (43) we get

$$\Sigma_{xx} = \frac{1}{3}\theta + \frac{K}{2\mu}\theta = \frac{2-\nu}{3(1-2\nu)}\theta \tag{44}$$

Recall that

$$\begin{aligned} K &= \frac{E}{3(1-2\nu)} \\ \mu &= \frac{E}{2(1+\nu)} \end{aligned}$$

Find the relation between θ and $\xi_{x,x}$

$$\Sigma_{xx} = \xi_{x,x} - \frac{1}{3}\theta \Rightarrow \theta = \frac{1-2\nu}{1-\nu}\xi_{x,x} \tag{45}$$

Then

$$T_{xx} = -K\theta - 2\mu\Sigma_{xx} = -\frac{E}{1-\nu^2}\xi_{x,x} \tag{46}$$

Equation of motion is

$$\rho\frac{\partial^2\xi_x}{\partial t^2} = T_{xx,x} + T_{xy,y} + T_{xz,z} = -\frac{E}{1-\nu^2}\frac{\partial^2\xi_x}{\partial x^2} \tag{47}$$

It gives the sound velocity

$$c_2 = \sqrt{\frac{E}{\rho(1-\nu^2)}}, \qquad \frac{c_1}{c_2} = \left(1-\nu^2\right)^{1/2} \tag{48}$$

0.5.3 Longitudinal waves along a rod embedded in incompressible medium.

Orient the x-axis of a Cartesian coordinate system along the rod. Incompressibility of the medium means that there is no strain in transverse direction, i.e. $\xi_y = \xi_z = 0$.

$$\Rightarrow \theta = \xi_{x,x}, \qquad \Sigma_{xx} = \xi_{x,x} - \frac{1}{3}\theta = \frac{2}{3}\xi_{x,x} \tag{49}$$

Using (49) we get

$$T_{xx} = -K\theta - 2\mu\Sigma_{xx} = -E\frac{1-\nu}{(1+\nu)(1-2\nu)}\xi_{x,x} \tag{50}$$

$$c_3 = \sqrt{\frac{E}{\rho}\frac{1-\nu}{(1+\nu)(1-2\nu)}}, \qquad \frac{c_1}{c_3} = \left(\frac{1-\nu}{(1+\nu)(1-2\nu)}\right)^{-1/2} \tag{51}$$

0.5.4 Shear waves in an extended solid.

The sound velocity has been calculated in BT. (BT-11.13)

$$c_4 = \sqrt{\frac{\mu}{\rho}} = \sqrt{\frac{E}{\rho 2(1+\nu)}} \tag{52}$$

$$\frac{c_1}{c_4} = [2(1+\nu)]^{1/2} \tag{53}$$

0.5.5 Torsional waves along a rod.

This has been also calculated in BT. (BT-11.28)

$$c_5 = c_4 = \sqrt{\frac{\mu}{\rho}}, \qquad \frac{c_1}{c_5} = [2(1+\nu)]^{1/2} \tag{54}$$