Solution for Chapter 14

(compiled by Xinkai Wu, modified by Jeff Atwell for 2004-2005 and 2006-2007 courses)
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A.

1. Exercise 14.1 part (b): Spreading of a laminar wake around a sphere [by Alexander Putilin/00]

Now with the cylinder replaced by the sphere, the cross section perpendicular to the flow is two dimensional and momentum conservation then implies:

$$\Delta v \cdot w^2 = \text{const}$$

namely $\Delta v \propto w^{-2}$.

The x component of the Navier-Stokes equation gives the same relation in the sphere case as in the cylinder case, $w \propto x^{1/2}$.

Combining these we get $\Delta v \propto x^{-1}$.

AND Exercise 14.4 part (b): Turbulent wake behind a sphere [by Alexei Dvoretskii/99]

The turbulent wake works in much the same way as its laminar counterpart, except that we should replace the intrinsic molecular viscosity ν with the kinematic turbulent viscosity $\nu_t \sim \Delta\bar{v}w$. The x component of the Navier-Stokes equation then gives the familiar relation

$$w \sim \left(\frac{\nu_t x}{V}\right)^{1/2} \sim \left(\frac{\Delta\bar{v}wx}{V}\right)^{1/2}$$

regardless of whether it's a cylinder or a sphere.

Now for the sphere, conservation of momentum implies that $\Delta\bar{v} \sim w^{-2}$.

Combining these we find $w \sim \text{const} \cdot x^{1/3}$. Using the fact that when $x \sim d$, $w \sim d$, we determine that const $\sim d^{2/3}$ and thus $w \sim d^{2/3}x^{1/3}$. Also we get $\Delta\bar{v} \sim \text{const} \cdot x^{-2/3}$. Using the fact that when $x \sim d$, $\Delta\bar{v} \sim V$, we can determine the const and find $\Delta\bar{v} \sim V\,(d/x)^{2/3}$.

2. Exercise 14.2 Spreading of a 2-dimensional laminar jet [by H.W. Lee and Kip Thorne]

This problem is pretty much parallel to the analysis in Section 13.4, except that the width w of the jet and its speed v_x now scale w.r.t. x differently from those in Section 13.4. (because now the ambient fluid is at rest and we have a nozzle ejecting fluid out.)

(a): This argument goes the same as that on Page 20 of Section 13.4. The Navier-Stokes equation reads

$$(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = \frac{-\boldsymbol{\nabla}P}{\rho} + \nu\boldsymbol{\nabla}^2\mathbf{v}$$

The y component of the N-S equation shows that the pressure difference $\Delta P \sim \rho v_x^2 w^2/x^2$. Recall that we'll use the x component of the N-S equation to find the velocity profile. So we plug the above expression for ΔP into the x component of the N-S equation and find there the ratio between the $\frac{\boldsymbol{\nabla} P}{\rho}$ term and the $(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}$ term (which is of the same order as the $\nu\nabla^2\mathbf{v}$ term) is $\sim \frac{w^2}{x^2} << 1$. Thus the pressure gradient term is indeed negligible for our purpose.

(b) The balance between the x components of the $(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}$ term and the $\nu\nabla^2\mathbf{v}$ term gives the familiar result

$$w \sim \left(\frac{\nu x}{v_x}\right)^{1/2}$$

Now the conservation of momentum along the x direction requires

$$v_x^2 w = \text{const}, \ \text{i.e.} \ \ v_x \sim w^{-1/2}$$

Combining these we find

$$w \sim x^{2/3}, \ \ v_x \sim x^{-1/3}$$

(c) Give the stream function the following trial form

$$\zeta = a x^p f(\xi)$$

where the normalization a and the index p are to be determined, and $f(\xi)$ is a function of the dimensionless number $\xi \equiv b x^{-2/3} y$ with $b \equiv \left(\frac{\mathcal{F}}{48\rho\nu^2}\right)^{1/3}$.

Then we find

$$\begin{aligned} v_x &= \frac{\partial \zeta}{\partial y} = a x^p f' \frac{\xi}{y} \\ v_y &= -\frac{\partial \zeta}{\partial x} = a x^{p-1}\left(\frac{2}{3}\xi f' - p f\right) \end{aligned}$$

Plugging these expressions into the x component of the N-S equation and throwing away terms subleading in the small parameter $\xi^2/(b^2 x^{2/3})$ (i.e. y^2/x^2, recalling that the jet is assumed to be very "thin"), we get

$$\frac{1}{3}a^2 b^2 x^{-\ \frac{7}{3}+2p}\left[(-2+3p)f'^2 - 3p f f''\right] = a b^3 \nu x^{-2+p} f'''$$

thus to have a self-similar solution we must satisfy

$$-\frac{7}{3} + 2p = -2 + p, \ \ \text{i.e.} \ \ p = \frac{1}{3}$$

and the N-S equation becomes

$$f''' + \frac{a}{3b\nu}(f'^2 + f f'') = 0$$

which can rewritten as

$$f''' + \frac{a}{3b\nu}(ff')' = 0$$

Integrating once we get

$$f'' + \frac{a}{3b\nu}ff' + C_1 = 0$$

We have the boundary conditions $v_y(y=0) = 0 \Rightarrow f(0) = 0$ and $\frac{\partial v_x}{\partial y}(y=0) = 0 \Rightarrow f''(0) = 0$, using which tells us $C_1 = 0$.

Integrating again gives

$$f' + \frac{a}{6b\nu}f^2 - C_2 = 0$$

solving which gives

$$f = \sqrt{\frac{6b\nu}{a}C_2}\ Tanh\left[\sqrt{\frac{C_2 a}{6b\nu}}(\xi + C_3)\right]$$

The boundary condition $f(0) = 0$ gives $C_3 = 0$. Using this result we find

$$v_x = (C_2 a)bx^{-1/3}Sech^2\left(\sqrt{\frac{C_2 a}{6b\nu}}\xi\right)$$

Now using the normalization condition

$$\mathcal{F} = \int_{-\infty}^{+\infty} \rho v_x^2 dy$$

we find

$$C_2 a = \left(\frac{3\mathcal{F}}{4\rho\sqrt{6\nu}}\right)^{2/3}\frac{1}{b}$$

which when plugged into the expression for v_x gives the final answer

$$v_x = \left(\frac{3\mathcal{F}^2}{32\rho^2\nu x}\right)^{1/3} Sech^2(\xi) = \left(\frac{3\mathcal{F}^2}{32\rho^2\nu x}\right)^{1/3} Sech^2\left(\left[\frac{\mathcal{F}}{48\rho\nu^2 x^2}\right]^{1/3} y\right)$$

AND Exercise 14.5 part (a): Spreading of a 2-dimensional turbulent jet [by H.W. Lee and Kip Thorne]

By now the following analysis should be very familiar to us:

x component of the N-S equation gives $w \sim \left(\frac{\nu_t x}{v_x}\right)^{1/2}$; Conservation of momentum gives $v_x \sim w^{-1/2}$; and we take $\nu_t \sim v_x w$. Combining these three facts we easily get

$$w \sim x, \quad v_x \sim x^{-1/2}$$

B.

Exercise 14.3 Reynolds stress and weak turbulence theory [by A. Dvoretskii/99]

(a) Let's write the Navier-Stokes equation

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla P + \nu\rho\nabla^2 \mathbf{v}$$

Decompose the velocity into a steady and a small fluctuating part

$$\mathbf{v} = \bar{\mathbf{v}} + \delta\mathbf{v}$$

And insert into the Navier-Stokes equation.

$$\rho \frac{\partial}{\partial t}\delta\mathbf{v} + \rho(\bar{\mathbf{v}} \cdot \nabla)\bar{\mathbf{v}} + \rho(\bar{\mathbf{v}} \cdot \nabla)\delta\mathbf{v} + \rho(\delta\mathbf{v} \cdot \nabla)\bar{\mathbf{v}} + \rho(\delta\mathbf{v} \cdot \nabla)\delta\mathbf{v} = -\nabla P + \nu\rho\nabla^2\bar{\mathbf{v}} + \nu\rho\nabla^2\delta\mathbf{v}$$

Taking the time average and using $\overline{\delta\mathbf{v}} = 0$ get

$$\rho(\bar{\mathbf{v}} \cdot \nabla)\bar{\mathbf{v}} = -\rho\overline{(\delta\mathbf{v} \cdot \nabla)\delta\mathbf{v}} - \nabla\bar{P} + \nu\rho\nabla^2\bar{\mathbf{v}}$$

The first term on the right-hand side can be rewritten as $-\nabla \cdot \mathbf{T}_R$ where $\mathbf{T}_R = \rho\overline{\delta\mathbf{v} \otimes \delta\mathbf{v}}$.

(b) To find the evolution of this tensor we take its time derivative

$$\frac{\partial \mathbf{T}_R}{\partial t} = \rho\overline{\frac{\partial \delta\mathbf{v}}{\partial t} \otimes \delta\mathbf{v}} + \rho\overline{\delta\mathbf{v} \otimes \frac{\partial \delta\mathbf{v}}{\partial t}}$$

Since $\frac{\partial \delta\mathbf{v}}{\partial t}$ involves averages of double products of velocity fluctuations, the time derivative of the velocity tensor will contain tensors that are time averages of triple products of velocity fluctuations. If we were to consider the time evolution of those tensors, because of the non-linearity of the equations, we'd have to cosider such tensors of higher and higher rank. To close the sequence it would be necessary to truncate it by specifying a priori the tensors of some rank.

(c) We can rewrite the time-averaged Navier-Stokes equation as

$$-\nabla\bar{P} = \rho\overline{(\delta\mathbf{v} \cdot \nabla)\delta\mathbf{v}} - \nu\rho\nabla^2\bar{\mathbf{v}} + \rho(\bar{\mathbf{v}} \cdot \nabla)\bar{\mathbf{v}}$$

and plug it back in into the full Navier-Stokes, note that $P = \bar{P} + \delta P$, equation (14.23) then follows immediately.

(d Multiplying by $\delta\mathbf{v}$ and taking the time average we get

$$\bar{\mathbf{v}} \cdot \nabla(\frac{1}{2}\rho\overline{\delta v^2}) + \mathbf{T}_R^{ij}\bar{v}_{i,j} + \nabla \cdot \overline{\left(\frac{1}{2}\rho\delta v^2\delta\mathbf{v} + \delta P\delta\mathbf{v}\right)} = \nu\rho\overline{\delta\mathbf{v} \cdot (\nabla^2\delta\mathbf{v})}$$

Regroup terms

$$\bar{\mathbf{v}} \cdot \nabla(\frac{1}{2}\rho\overline{\delta v^2}) + \nabla \cdot \overline{\left(\frac{1}{2}\rho\delta v^2\delta\mathbf{v} + \delta P\delta\mathbf{v}\right)} = \nu\rho\overline{\delta\mathbf{v} \cdot (\nabla^2\delta\mathbf{v})} - \mathbf{T}_R^{ij}\bar{v}_{i,j}$$

The terms on the left hand side are the convective time derivative and the divergence of the flow of turbulent energy density typical of conservation laws. On the right hand side are possible sources of energy or its dissipation. In this case the first term is energy dissipation due to molecular viscosity and the second term is due to energy exchange between the ordered and turbulent motion.

(v) This can be seen if we take the Navier-Stokes equation and perform a similar transformation to get the law of ordered motion energy conservation

$$\nabla \cdot ((\frac{1}{2}\rho\bar{v}^2)\bar{\mathbf{v}}) + \nabla \cdot (P\bar{\mathbf{v}}) = \nu\rho\bar{\mathbf{v}} \cdot \nabla^2\bar{\mathbf{v}} - \bar{\mathbf{v}} \cdot \nabla\mathbf{T}_R$$

For incompressible fluid the full divergence $\nabla \cdot (\bar{\mathbf{v}}\mathbf{T}_R) = 0$ and so we can rewrite

$$\nabla \cdot ((\frac{1}{2}\rho\bar{v}^2)\bar{\mathbf{v}}) + \nabla \cdot (P\bar{\mathbf{v}}) = \nu\rho\bar{\mathbf{v}} \cdot \nabla^2\bar{\mathbf{v}} + \mathbf{T}_R^{ij}\bar{v}_{i,j}$$

We see that, indeed, the last term describes the exchange of energy between ordered and turbulent motion.

C.

1. Exercise 14.7 Excitation of earth's normal modes by atmospheric turbulence [by Alexander Putilin/00]

(a) (i) Let's first do this via dimensional analysis. Pressure has the dimension of ρv^2. Thus we just need to use q and f to construct a quantity with dimension v^2. Noting that the dimension of q is length^2/time^3, and that of f is 1/time. Requiring

$$v^2 \sim \frac{\text{length}^2}{\text{time}^2} \sim q^\alpha f^\beta$$

we can solve for the indices α and β and find them to be $\alpha = 1$, $\beta = -1$. Thus we see

$$P(f) \sim \frac{\rho q}{f}$$

(ii) Now let's use the eddy size and speed analysis. $P \sim \rho v(k)^2$. Recall B.T. eqn. (14.19) gives $v(k) \sim (q/k)^{1/3}$, and we also know that $f \sim v(k)k$. Combining the above two facts we get $k \sim q^{-1/2}f^{3/2}$ and $v \sim (q/f)^{1/2}$. So we finally get $P \sim \rho q/f$.

(b) If we also take dissipation (due to viscosity) into account, then besides the energy cascade rate q we will have another dimensioned constant: the energy dissipation rate $S \sim (\text{length/time})^3$. Then using dimensional analysis to construct P, we need $P \sim \rho v^2 \sim \rho q^\alpha f^\beta S^\gamma$ with

$$v^2 \sim (\text{length/time})^2 \sim q^\alpha f^\beta S^\gamma$$

we see that the solution isn't unique (two equations for the three indices α, β, γ). To have $\beta = -2/3$(i.e. $P(f) \sim 1/f^{2/3}$), we just need to set $\alpha = 2/3, \gamma = 2/9$.

(c) The eddy viscosity: $\nu_t \sim \frac{1}{3} v_l l$. l is the length scale of the largest eddies, $l \sim 5km$. v_l is turnover velocity of the largest eddies, $v_l \sim \pi l f_{min}$, $f_{min} \sim 0.5mHz$. This gives $\nu_t \sim l^2 f_{min} \sim 10^4 \frac{m^2}{s}$. The molecular viscosity is $\nu \sim 10^{-5} \frac{m^2}{s}$, so $\nu_t/\nu \sim 10^9$, some 9 orders of magnitude more.

The cascading energy per unit area per unit time $W_{turb} \sim \rho q H$, where $H \sim 10km \sim$height of atmosphere.

$q \sim v_l^3 l^{-1} \sim 2m^2/s^3$, thus $W_{turb} \sim 2 \times 10^4 erg/cm^2 \cdot s$.

This energy dissipates into heat at the smallest lengthscale, so the fraction of solar energy required to maintain the turbulence is

$$\frac{W_{turb}}{W_{solar}} \sim 2\%$$

(d From part (a), $k^{-2/3} \sim \tau q^{1/3}$, or $x^{2/3} \sim f^{-1} q^{1/3}$, where x is the characteristic spatial scale of the pressure fluctuations, and f is the frequency. This gives $x \sim f^{-3/2}$

We know that at minimal frequency $f_{min} \sim 0.5mHz$, $x \sim l \sim 5km$. Then at $f \sim 1mHz$, $x \sim 5km \cdot 2^{-3/2} \sim 2km$.

The characteristic spatial scale of the normal modes is their wavelength $\lambda \sim \frac{\text{wave speed}}{f} \sim 5 \times 10^3 km$.

On the surface area λ^2, there are $N = (\lambda/x)^2$ roughly independent regions of turbulent pressure, thus the pressure fluctuation is reduced by the factor $1/\sqrt{N} \sim x/\lambda \sim 1/2500$, so the averaged pressure is $P \sim \Delta P \frac{x}{\lambda} \sim P(f)\frac{x}{\lambda}$. At $f = 1mHz$: $P(f) \sim \frac{\rho q}{f} \sim 2Pa$, thus $P \sim 10^{-3} Pa$.

(v) Assume that fluctuating turbulence pressure excites the mode with resonant frequency $f \sim 1mHz$ and resonance parameter $q \sim \frac{\text{ring down time}}{period} \sim \frac{\text{a few days}}{10^3 s} \sim 100$.

The force due to the pressure is "remembered" by the mode for $\sim q$ oscillations and since it adds up randomly the amplitude of the oscillation can be estimated as

$$\xi \sim \lambda \frac{P}{\mu} \sqrt{q}$$

where $P = \text{rms pressure} \approx 10^{-3} Pa$ (calculated in part (d)), $\mu \sim 10^{12} dyne/cm^2$, and $\lambda \sim 5 \times 10^3 km$ is the wavelength. P/μ gives the characteristic strain due to the pressure P. And the acceleration is $a \sim (2\pi f)^2 \lambda \sim 10^{-9} cm/s^2$.

To derive more accurate estimate we should remember that the turbulent pressure excites not just one mode, but a number of modes in some frequency range around f. It gives smaller value of a, closer to the experimental one.

2. Exercise 14.8 Effect of drag [by R.D.Blandford]

$$R = \frac{du}{\nu} \approx 100(\frac{d}{1\text{mm}})(\frac{u}{1\text{ms}^{-1}})$$

When $R > 10$ we can assume that the drag coefficient
$C_0 \approx 1$ and so
$F_{\text{drag}} \approx \rho u^2 d^2$.
In absense of drag the range is $D \approx \frac{u^2}{g}$.
When drag dominates $F_{\text{drag}} D \approx m u^2$ and therefore $D \approx \frac{m}{\rho d^2}$. Therefore, we can define critical velocity $u_c = (\frac{mg}{\rho d^2})^{\frac{1}{2}}$. For velocities higher than u_c the drag is comparable to gravity.
The table below compares the different types of balls

Ball	m(g)	d(mm)	u(m/s)	R	u_c(m/s)
Golf	46	43	60	2.6×10^5	16
Baseball	140	75	40	3×10^5	16
T. Tennis	2.5	38	13	5×10^4	4

We see that drag is very important in all the cases when the ball is hit hard.

D.
1. Exercise 14.10 Feigenbaum sequence [by Kip Thorne]

$$x_{n+1} = 4a x_n (1 - x_n)$$

where $0 \leq a \leq 1$ and $0 \leq x \leq 1$

Range	converges to	critical value
$0 \leq a \leq 1/4$	stable point $x^* = 0$	
$1/4 \leq a \leq 3/4$	stable point $x^* = 1 - \frac{1}{4a}$	$a_1 = \frac{3}{4}$
$3/4 \leq a \leq 0.862$	period 2	$a_2 = 0.862...$
$0.862 \leq a \leq 0.885$	period 4	$a_3 = 0.885...$
$0.885 \leq a \leq 0.890$	period 8	$a_2 = 0.890...$
$0.892 \leq a$	period ∞ chaos sets in	$a_2 = 0.892...$

The values of $\alpha_j = \frac{a_j - a_{j-1}}{a_{j+1} - a_j}$ are:

$\alpha_1 = 4.5$, $\alpha_2 = 4.8$, $\alpha_3 = 4.6$, the limit is 4.669201

2. Exercise 14.12 Strange attractors [by Kip Thorne]
(a)

$$x_{n+1} = a(1 - 2|x_n - \frac{1}{2}|),\ \ a > 0$$

$$x_{n+1} = \begin{cases} 2a(1 - x_n), & x_n \geq \frac{1}{2} \\ 2ax_n, & x_n < \frac{1}{2} \end{cases}$$

We immediately see that $x_{n+1} \leq a$ for all $n = 1, 2....$

A) $a < \frac{1}{2}$ case:
For $n \geq 2$, $x_n \leq a < \frac{1}{2}$. Hence
$x_{n+1} = 2ax_n$. This is a geometric sequence that converges to zero which is the stable fixed point.

B) $a > \frac{1}{2}$ case:
We have a fixed point x_p satisfying $x_p = 2a(1 - x_p)$ or $x_p = \frac{2a}{1+2a}$. To test the stability of the fixed point, let $\epsilon > 0$ be a small positive number. Since $x_p > \frac{1}{2}$ then $x_p + \epsilon > \frac{1}{2}$.

$$x_{n+1} = 2a(1 - x_p - \epsilon) = x_p - 2a\epsilon$$

$$x_{n+k} = x_p + (-1)^k (2a)^k \epsilon$$

So the deviations grow and the fixed point is unstable.
Because $x_{n+1} \leq a$, we have $2a(1 - x_n) > 2a(1 - a)$ and hence

$2a(1 - a) \leq x \leq a.$

(b) $a_{\text{crit}} = \frac{1}{2}$, for $a = 0.8$
$x_{\text{min}} = 0.32$
$x_{\text{max}} = 0.8$

(c) Numerical calculations show that $n(\epsilon) \sim -\log_2(\epsilon)$ is satisfied.