Solution for Problem Set 16A

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1 A. Transonic Flows

1.1 Problem 16.4 - Adiabatic, Spherical Accretion of Gas Onto a Black Hole

[by Guodong Wang/03]

(a). There are four dimensional variables M, ρ_∞, c_∞, G with only three independent dimensional fundamentals: mass, length and time. So by the dimensional considerations alone we cannot get a unique answer. Considering $\dot{M}$ should be proportional to ρ_∞, then

$$\dot{M} = K(\gamma)\frac{\rho_\infty (GM)^2}{c_\infty^3}, \tag{1}$$

Where $K(\gamma)$ is a dimensionless function of γ.

(b). The mass accretion rate $\dot{M}$ at radius r is $4\pi r^2 \rho v$. By mass conservation, we know it is a constant at all radii. Calculating $\dot{M}$ at $r \sim \frac{GM}{c_\infty^2}$, where it is reasonable to expect $v \sim c_\infty$:

$$\dot{M} \sim 4\pi\rho_\infty c_\infty (\frac{GM}{c_\infty^2})^2 \sim K(\gamma)\frac{\rho_\infty (GM)^2}{c_\infty^3}. \tag{2}$$

(c). The gas will finally stop on the surface of the neutron star so the flow speed has to change to subsonic near the surface. For the black hole, there is nothing to resist the flow, so it remains supersonic all the way through the horizion. Because no sound waves can propagate upstream in the supersonic region, there is no way for the flow before the shock to know that there will be a shock; so that flow is the same for the neutron star as for the black hole. Since the mass accretion rate is determined by conditions at the sonic point, which is before the shock, it must also be the same in both case.

(d). The Euler equation for $v(r)$ is

$$\rho v \frac{dv}{dr} = -\frac{dP}{dr} - \frac{GM\rho}{r^2} \tag{3}$$

The mass conservation reads

$$\frac{d\dot{M}}{dr} = 4\pi \frac{d}{dr}(r^2 \rho v) = 0 \tag{4}$$

Plugging in $c^2 = (\partial P/\partial \rho)_s$,

$$\frac{dP}{dr} = c^2 \frac{d\rho}{dr} \tag{5}$$

Inserting eq. (4) and eq. (5) into eq. (3), we get

$$(v^2 - c^2)\frac{1}{\rho}\frac{d\rho}{dr} = \frac{GM}{r^2} - \frac{2v^2}{r}. \tag{6}$$

So at the sonic point,

$$v_s^2 = c_s^2 = \frac{GM}{2r_s} \tag{7}$$

(e). Put eq. (7) into Bernoulli equation

$$\frac{1}{2}v_s^2 + \frac{v_s^2}{\gamma - 1} - \frac{GM}{r_s} = \frac{c_\infty^2}{\gamma - 1} \tag{8}$$

thus

$$c_s^2 = \frac{2c_\infty^2}{5 - 3\gamma}, \qquad r_s = \frac{(5 - 3\gamma)}{4}\frac{GM}{c_\infty^2}. \tag{9}$$

Calculate $\dot{M}$ at r_s (since it is constant at all r) and note $\rho^{1-\gamma}c^2 = \rho_\infty^{1-\gamma}c_\infty^2$ (BT-16.8). We get

$$\dot{M} = \frac{4\pi\lambda G^2 M^2 \rho_\infty}{c_\infty^3}, \tag{10}$$

where

$$\lambda = (\frac{1}{2})^{\frac{\gamma+1}{2(\gamma-1)}}(\frac{5-3\gamma}{4})^{\frac{3\gamma-5}{2(\gamma-1)}} \tag{11}$$

For $\gamma = 5/3$, $\lambda \simeq 0.25$ (by taking limit). So this factor is of order 1. It agrees well with $\dot{M}$ in parts (a) and (b).

(f). Let $\gamma = 5/3$, $c_\infty = \sqrt{\gamma P/\rho} = \sqrt{\gamma nkT/\rho} \simeq 10^5$ m/s, solar mass $\sim 2 \times 10^{33}$ g, Then $\dot{M} \sim 10^{10}$ g/s. Integrating eq. (10), to double the mass, it takes

$M/2\dot{M} \simeq 10^{24}s \sim 3 \times 10^{16}$ year. So it is too slow to double the hole's mass by this way in the age of the universe (1.5×10^{10} year).

2 B. Riemann invariants

2.1 Problem 16.6 - Riemann Invariants for Shallow-Water Flow

[by Guodong Wang/03]

(a).

$$\frac{\partial h}{\partial t} + \frac{\partial (hv)}{\partial x} = 0 \tag{12}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + g\frac{\partial h}{\partial x} = 0 \tag{13}$$

(13)$\pm\sqrt{\frac{g}{h}}$ (12),
we get

$$\frac{\partial (v \pm 2\sqrt{gh})}{\partial t} + (v \pm \sqrt{gh})\frac{\partial (v \pm 2\sqrt{gh})}{\partial x} = 0 \tag{14}$$

So the Riemann invariants are

$$J_\pm \equiv v \pm 2\sqrt{gh} \tag{15}$$

with the characteristic speeds

$$V_\pm \equiv v \pm \sqrt{gh} \tag{16}$$

The conresponding conservation equations are

$$(\frac{\partial}{\partial t} + V_\pm \frac{\partial}{\partial x})J_\pm = 0 \tag{17}$$

(b). The argument is the same as in section 16.4.1 of the text. Here $v = \sqrt{2gh} - constant$, so the water at the peak of the wave moves faster than the water in the bottom. This causes the leading edge of the wave to steepen.

(c).

$$J_+ = v + 2\sqrt{gh} = 2\sqrt{gh_0} \tag{18}$$

As in Eq. (16.40c) of the text,
solving for the leftward characteristics C_-, we obtain

$$C_- : \qquad x = (v - \sqrt{gh})t = (2\sqrt{gh_0} - 3\sqrt{gh})t \tag{19}$$

So

$$h(x,t) = \frac{h_0}{9}(2 - \frac{x/t}{\sqrt{gh_0}})^2 \tag{20}$$

and

$$v(x,t) = \frac{2}{3}(x/t + \sqrt{gh_0}) \tag{21}$$

at $-\sqrt{gh_0}t < x < 2\sqrt{gh_0}t$. The water is unperturbed outside this range.