

Solution for Problem Set 16B&18A

(compiled by Guodong Wang, revised by Geoffrey Lovelace for the 2004-2005 Academic Year and by Kip for 2006-2007)

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A. Shocks

Problem 16.7 - Hydrolic jumps and breaking ocean waves

[by Kip Thorne/03, revised by Geoffrey Lovelace]

(a). Behind the wave, at height z ,

$$P_2 = \rho g(h_2 - z) \quad (1)$$

and ahead the wave, under the water the pressure is

$$P_1 = \rho g(h_1 - z) \quad (2)$$

Notice that the densities are the same on either side of the shock; this follows from the fact that the water is incompressible.

Analyze the flow in the rest frame of the breaking wave (the "shock wave"), so the speed of upstream water (water closes to the breaker) is $v_1 = v_{break}$. Let v_2 be the downstream flow speed away from the breaker. Then

$$\rho h_2 v_2 = \rho h_1 v_1, \quad (3)$$

and momentum conservation is

$$\rho v_1^2 h_1 + \int_0^{h_1} P_1 dz = \rho v_2^2 h_2 + \int_0^{h_2} P_2 dz \quad (4)$$

Evaluating the integrals gives

$$v_1^2 h_1 + \frac{1}{2} g h_1^2 = v_2^2 h_2 + \frac{1}{2} g h_2^2 \quad (5)$$

Equation (3) suggests that we define the dimensionless quantity

$$x \equiv \frac{h_1}{h_2} = \frac{v_2}{v_1} \quad (6)$$

The quantity x is positive and (as can be seen experimentally) less than unity.

Rewriting Eq. (5) in terms of x yields

$$h_1 v_1^2 x^3 - \frac{1}{2} h_1 (g h_1 + 2 v_1^2) x^2 + \frac{g h_1^2}{2} = 0. \quad (7)$$

The three solutions of this cubic equation are¹

$$x = 1 \text{ and } x = \frac{1}{4} \left(y \pm \sqrt{y(8+y)} \right),$$

$$y \equiv \frac{gh_1}{v_1^2}. \quad (8)$$

The first solution is trivial; there is no shock. Choosing the negative sign in the second solution implies that x is negative, which is unphysical. Therefore,

$$x = \frac{1}{4} \left(y + \sqrt{y(8+y)} \right). \quad (9)$$

Notice that x is positive. In terms of this quantity,

$$h_2 = \frac{h_1}{x} \text{ and } v_2 = v_1 x. \quad (10)$$

b. Mathematically, we have only two physical quantities to solve for, the horizontal velocity and height of the flow. Therefore, we should only need two equations (in this case, momentum conservation and mass conservation).

More physically, energy dissipation by the shock is not relevant to the motion of the water. For an ideal gas, energy dissipation causes the temperature of the gas to change as it passes through the shock, which in turn causes the density of the gas to change. However, the density, height, and flow speed of the water do not vary significantly due to the (small) temperature change across the shock. Therefore, you expect that energy conservation is not important when considering junction conditions for the shock.

c. To show that the flow is supersonic in region 1 and subsonic in region 2, it is convenient to solve for v_1 and v_2 rather than for v_2 and h_2 (as we did in part a. This solution is straightforward to obtain; the result is

$$v_1 = \sqrt{\frac{gh_2(h_1+h_2)}{2h_1}} = \sqrt{gh_1} \sqrt{\frac{h_2}{h_1} \frac{(h_1+h_2)}{2h_1}} \quad (11)$$

$$v_2 = \sqrt{\frac{gh_1(h_1+h_2)}{2h_2}} = \sqrt{gh_2} \sqrt{\frac{h_1}{h_2} \frac{(h_1+h_2)}{2h_2}}. \quad (12)$$

The flow is subsonic in region 2 (downstream) and supersonic in region 1 (upstream) provided that the height in region 2 is greater than the height in region 1.

This can be seen experimentally. If you hold a plate under a kitchen tap with a high flow rate, you can see that water in the upstream region has a lower height than water in the downstream region. Thus, the flow is subsonic downstream and supersonic upstream.

¹The dimensionless quantity y must be positive, since g , h_1 , and v_1 are all positive. I use this fact to simplify these solutions.

(d). In part a, we noticed that the velocity of the shock is v_1 . Therefore,

$$v_{break} = v_1 = \sqrt{\frac{gh_2(h_1 + h_2)}{2h_1}}. \quad (13)$$

Problem 16.11 - Relativistic Shock

[by Geoffrey Lovelace]

a. Begin with the stress tensor of a perfect fluid² \mathbb{T} is given by Eq. (1.154) of the text:

$$\mathbb{T} = P\mathbf{g} + (\rho + P)\mathbf{u} \otimes \mathbf{u} \quad (14)$$

Here, ρ and P are the density and pressure of the fluid when the fluid is at rest. In index notation,

$$T^{\mu\nu} = Pg^{\mu\nu} + (\rho + P)u^i u^j. \quad (15)$$

The energy flux is (in some frame)

$$T^{0j} = (\rho + P)u^i u^j. \quad (16)$$

The momentum flux is

$$T^{ij} = P\delta^{ij} + (\rho + P)u^i u^j \quad (17)$$

Now move to the local Lorentz frame at some point in the shock. Choose coordinates so that the shock is in the yz plane, while the fluid flows in the x direction (i.e., normal to the shock plane) with a four velocity with components $(u^0, u^x, u^y, u^z) = (\gamma, u, 0, 0)$ where $u = \gamma v$, with $\gamma \equiv \sqrt{1 - v^2}$. Then the conservation laws for energy, momentum, and rest mass are

$$[T^{0x}] = [(\rho_R + P)\gamma u] = 0 \quad (18)$$

$$[T^{xx}] = [P + (\rho_R + P)u^2] = 0 \quad (19)$$

$$[j] = [mnu] = 0. \quad (20)$$

Here, j is the rest mass flux flux (the particles of the fluid each have mass m), n is the number density of particles in the fluid, and I have adopted the notation of Chapter 16 that says the relativistic density is ρ_R . (As in the text, “[]” means “difference across the boundary.”)

The only work that remains is to recast these equations in the form of Eqs. (16.48). The volume per rest mass is related to the number density by

$$mn = \rho_o = (1/V) \Rightarrow V = \frac{1}{\rho_o}. \quad (21)$$

Here, ρ_o is the rest mass density of the fluid. Rewriting rest mass conservation in terms of V yields

$$\left[\frac{u}{V}\right] = 0 \Rightarrow u_1 = \gamma_1 v_1 = jV_1 \text{ and } u_2 = \gamma_2 v_2 = jV_2. \quad (22)$$

²Just as in Sec. 16.5 of the text, I assume that the fluid is ideal.

This is Eq. (16.48c).

Next, recall that $\rho_R = \rho_o(1 + \epsilon)$, where ϵ is the internal energy per unit rest mass. (I use the symbol ϵ to avoid confusion with the four velocity component u defined above.) From this, define $\eta = \rho_o^{-1}(\rho_R + P) = 1 + \epsilon + P/\rho_o = 1 + h$; this is the relativistic enthalpy per unit rest mass.

Momentum conservation can be rewritten in terms of η , V , j , and P :

$$[P + (\rho_o \eta)j^2 V^2] = 0 \Rightarrow j^2 = \frac{P_2 - P_1}{\rho_o(\eta_1 V_1^2 - \eta_2 V_2^2)} = \frac{P_2 - P_1}{\eta_1 V_1 - \eta_2 V_2}. \quad (23)$$

This is Eq. (16.48b). In the last line, I used the identity $\rho_o V = 1$.

Finally, rewrite energy conservation:

$$[\rho_o \eta \gamma j V] = 0 = [\eta \gamma]. \quad (24)$$

This is not yet in the form of Eq. (16.48a). To get there, divide the momentum conservation equation by $(\eta_1 V_1 + \eta_2 V_2)$ to obtain

$$\frac{1}{\eta_1 V_1 + \eta_2 V_2} = \frac{P_2 - P_1}{(\eta_1 j V_1)^2 - (\eta_2 j V_2)^2} = \frac{P_2 - P_1}{(\eta_1 u_1)^2 - (\eta_2 u_2)^2}. \quad (25)$$

Thus,

$$(\eta_1 u_1)^2 - (\eta_2 u_2)^2 = (P_2 - P_1)(\eta_1 V_1 + \eta_2 V_2). \quad (26)$$

Squaring the energy conservation equation yields

$$(\eta_2 \gamma_2)^2 - (\eta_1 \gamma_1)^2 = 0. \quad (27)$$

Adding the two previous lines finally gives

$$\begin{aligned} \eta_2^2(\gamma_2^2 - u_2^2) - \eta_1^2(\gamma_1^2 - u_1^2) &= (P_2 - P_1)(\eta_1 V_1 + \eta_2 V_2) \\ \Rightarrow \eta_2^2 - \eta_1^2 &= (P_2 - P_1)(\eta_1 V_1 + \eta_2 V_2). \end{aligned} \quad (28)$$

In the last equality, I used the identity $\gamma^2 - u^2 = 1$. This is Eq. (16.48a).

b. The relativistic equations are very similar to the non-relativistic equations. You should be able to convince yourself that Eqs. (16.48) and (16.42) have exactly the same form provided ηV is taken to be the “relativistic specific volume,” γv is taken to be the relativistic velocity, and $1 + h$ is taken to be the relativistic enthalpy. Therefore, a relativistic P vs. ηV plot looks the same as a non-relativistic P vs. V plot, provided that you replace the enthalpy with the relativistic enthalpy (and the velocity with the relativistic velocity) when computing the shock adiabat.

The interpretation of the relativistic P vs. ηV plot then follows. Choose an initial point in $P - \eta V$ space as the upstream point. Possible downstream points must also lie on this adiabat (according to the relativistic equation of state of the fluid). Possible downstream points have higher entropy than the upstream point (curves of constant entropy also depend on the particular equation of state chosen). Finally, Eq. (16.48b) describes a straight line on the P vs. ηV plot;

this line of constant rest mass flux must connect the upstream point with the downstream point.

c. Begin with rest mass conservation. In the nonrelativistic limit, $\gamma \rightarrow 1$. Also, since the rest mass dominates the energy contribution, $\eta \rightarrow 1$. Thus, Eqs. (16.48) reduce to

$$\begin{aligned} (1 + 2h_2) - (1 + 2h_1) &= (P_2 - P_1)(V_1 + V_2) \\ \Rightarrow h_2 - h_1 &= \frac{1}{2}(P_2 - P_1)(V_1 + V_2) \\ \Rightarrow \epsilon_2 - \epsilon_1 &= \frac{1}{2}(P_2 + P_1)(V_1 - V_2) \end{aligned} \quad (29)$$

$$j^2 = \frac{(P_2 - P_1)}{V_1 - V_2} \quad (30)$$

$$v_1 - v_2 = j(V_2 - V_1) = \sqrt{(P_2 - P_1)}\sqrt{V_1 - V_2}. \quad (31)$$

These are Eqs. (16.42).

B. Similarity Solutions

Problem 16.12 - Underwater Explosions

[by Alexei Dvoretzkii/00]

Note from Geoffrey Lovelace: In 0416.2.K of Chapter 16, f , g , and h are written as \hat{P} , $\hat{\rho}$, and \hat{v} , respectively.

(i) Equation of continuity

$$\partial_t \rho + \frac{2}{r}(\rho v) + \partial_r(\rho v) = 0$$

$$\rho = \frac{\gamma + 1}{\gamma - 1} \rho_0 g$$

$$v = \frac{2}{\gamma + 1} \dot{R} h$$

Rewriting the equation in terms of ξ and R obtain

$$\left(\frac{\gamma + 1}{\gamma - 1}\right) \left(-\frac{\xi \dot{R}}{R}\right) \rho_0 g' + \frac{2}{\xi R} \frac{2}{(\gamma - 1)} \rho_0 \dot{R} g h + \frac{1}{R} \partial_\xi \left(\frac{2}{\gamma - 1} \rho_0 \dot{R} g h\right) = 0$$

Simplifying

$$-\xi g' + \frac{4}{(\gamma + 1)} \frac{1}{\xi} g h + \frac{2}{(\gamma + 1)} \partial_\xi (g h) = 0$$

So far, complete generality has been retained. Now let's take

$h = \xi$.

$$-\xi g' + \frac{4}{(\gamma+1)}g + \frac{2}{(\gamma+1)}g'\xi + \frac{2}{(\gamma+1)}g = 0$$

This is readily solved to give

$$g = \xi^{\frac{6}{\gamma-1}},$$

where the integration constant has been fixed by the demand that $g = 1$ at $\xi = 1$.

We see that for $\gamma = 7$ the solution takes a simple form

$$g = \xi$$

(ii) Equation of motion

$$\partial_t v + v \partial_r v + \frac{1}{\rho} \partial_r P = 0$$

$$P = \frac{2}{\gamma+1} \rho_0 \dot{R}^2 f$$

Rewriting the equation in terms of ξ and R obtain

$$-\frac{\xi \dot{R}}{R} \partial_\xi \left(\frac{2}{\gamma+1} \right) \dot{R} h + \dot{R} \partial_R \left(\frac{2}{\gamma+1} \right) \dot{R} h + \left(\frac{2}{\gamma+1} \right) \dot{R} h \frac{1}{R} \partial_\xi \left(\frac{2}{\gamma+1} \right) \dot{R} h + \frac{\gamma-1}{\gamma+1} \rho_0^{-1} g^1 \frac{1}{R} \partial_\xi \left(\frac{2}{\gamma+1} \right) \rho_0 \dot{R}^2 f = 0$$

Using $\partial_R \dot{R} = -\frac{3}{2} \frac{\dot{R}}{R}$ get

$$-\xi h' \frac{2}{\gamma+1} + h \frac{2}{\gamma+1} \left(-\frac{3}{2} \right) + \frac{4}{(\gamma+1)^2} h h' + 2 \frac{\gamma-1}{(\gamma+1)^2} \frac{f'}{g} = 0$$

Substituting $h = \xi$ and $g = \xi$ and simplifying get

$$f' = 3\xi^2$$

And therefore

$$f = \xi^3$$

(iii) Entropy equation

Using

$$\partial_t + v \partial_r = -\frac{3}{4} \frac{\dot{R}}{R} \xi \partial_\xi + \dot{R} \partial_R$$

and

$$\frac{P}{\rho^\gamma} \propto \dot{R}^2 \xi^{-4}$$

it is straightforward to verify that the above solution also satisfies the entropy equation.

(iv) It's trivial to take the integral to get

$$E = \frac{2\pi R^5 \rho_0}{225t^2}$$

(v) The Sedov-Taylor solution will be more or less valid until the velocity of the shock front \dot{R} falls to about the speed of sound in water.

$$\dot{R} \approx \left(\frac{E}{\rho_0 R^3}\right)^{\frac{1}{2}}$$

so it will happen when

$$R \approx \left(\frac{E}{\rho_0 c_0^2}\right)^{\frac{1}{3}}$$

Using $E = 10^{10}\text{J}$, $\rho_0 = 10^3\text{kgm}^{-3}$ and $c_0 = 1.5 \times 10^3\text{ms}^{-1}$ get

$$R \approx 20\text{m}$$

Problem 16.14 - Similarity Solution for Shock Tube

[by Guodong wang/03]

Let $\xi = \frac{x}{c_0 t}$. then $\frac{\partial}{\partial t} = -\frac{x}{c_0 t^2} \frac{d}{d\xi}$ and $\frac{\partial}{\partial x} = \frac{x}{c_0 t} \frac{d}{d\xi}$. Inserting these relations into the two partial differential equations (BT-16.32) with $\int dP/\rho c = 2c/(\gamma-1)$,

$$\left(\frac{\partial}{\partial t} + (v \pm c) \frac{\partial}{\partial x}\right)(v \pm \frac{2c}{\gamma-1}) = 0 \quad (32)$$

becomes a pair of ordinary differential equations,

$$((v - c_0 \xi) \pm c)(v' \pm \frac{2c'}{\gamma-1}) = 0 \quad (33)$$

Then it is straightforward to get the non-trivial solutions,

$$c' = -\left(\frac{\gamma-1}{\gamma+1}\right)c_0 \Rightarrow c = \text{Constant} - \left(\frac{\gamma-1}{\gamma+1}\right)\frac{x}{t} \quad (34)$$

$$v = c_0 \xi + c \quad (35)$$

The integration constant can be determined by matching v and c at the boundary with the unperturbed fluid: $v \rightarrow 0$, $c \rightarrow c_0$ when $-x/t \rightarrow c_0$. we thereby obtain

$$c = \frac{2c_0}{\gamma+1} - \left(\frac{\gamma-1}{\gamma+1}\right)\frac{x}{t}, \quad v = \frac{2}{\gamma+1}\left(c_0 + \frac{x}{t}\right) \quad (36)$$

at $-c_0 t < x < \frac{2c_0}{\gamma-1} t$.

C. MHD

Exercise 18.2 Diffusion of Magnetic Field

[by Xinkai Wu/02]

As we assume the plasma has sufficient inertia to remain at rest, we can set $\mathbf{v} = \mathbf{0}$ in all the equations.

(a) The sum of the rate of change of the magnetic energy and that of heat production via Ohmic heating is given by

$$\frac{\partial}{\partial t} \frac{\mathbf{B}^2}{2\mu_0} + \mathbf{j} \cdot \mathbf{E} = \frac{1}{\mu_0^2 \kappa_e} [\mathbf{B} \cdot \nabla^2 \mathbf{B} + (\nabla \times \mathbf{B}) \cdot (\nabla \times \mathbf{B})] \quad (37)$$

where we have used equation (18.9) for $\frac{\partial \mathbf{B}}{\partial t}$ and equations (18.6) and (18.7) for \mathbf{j} and \mathbf{E} , respectively.

Now writing out the components in Cartesian coordinates explicitly and using $\nabla \cdot \mathbf{B} = 0$, we readily reduce this sum to

$$\frac{1}{\mu_0^2 \kappa_e} \partial_k [B_j (\partial_k B_j - \partial_j B_k)] = -\nabla \cdot \left(\frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right) \quad (38)$$

which is a total derivative and vanishes upon integration over the whole space, namely the reduction of magnetic energy as the field decays is compensated by the Ohmic heating of the plasma.

(b) The magnetic field satisfies the diffusion equation:

$$\frac{\partial B_z}{\partial t} - \frac{1}{\mu_0 \kappa_e} \nabla^2 B_z = 0 \quad (39)$$

and the initial condition is

$$B_z(t=0) = B_0 \text{ for } \varpi < R, \text{ and } 0 \text{ elsewhere} \quad (40)$$

with R being the radius of the cylinder. The boundary condition is given by

$$B_z(\varpi = R) = B_z \text{ outside} = 0; \quad B_z(\varpi = 0) = \text{finite} \quad (41)$$

where we have used the fact that since there is no surface current (we've turned off the solenoid current that generated the initial magnetization) B_z is continuous across the boundary. Also we approximate the magnetic field outside the cylinder as zero because we assume that the decay is slow (as a consequence of large κ_e) and thus EM radiation into the vacuum can be neglected.

Solving the above boundary value problem is straightforward (see e.g. Mathews and Walker), and we find

$$B_z(\varpi, t) = \sum_{i=1}^{\infty} c_i e^{-\frac{\xi_i^2}{\mu_0 \kappa_e R^2} t} J_0\left(\frac{\xi_i \varpi}{R}\right) \quad (42)$$

with $J_0(\xi)$ being the zeroth Bessel function, and ξ_i being the location of its i th zero, $0 < \xi_1 < \xi_2 < \dots$. So for large time, when the field has decayed to a small fraction of its original value, all the higher (i.e. $i > 1$) modes have decayed away, and we have

$$B_z(\varpi, t) \approx c_1 e^{-\frac{\xi_1^2}{\mu_0 \kappa_e R^2} t} J_0\left(\frac{\xi_1 \varpi}{R}\right) = 1.6 B_0 e^{-\frac{2.4^2}{\mu_0 \kappa_e R^2} t} J_0\left(\frac{2.4 \varpi}{R}\right) \quad (43)$$

where we have used $\xi_1 \approx 2.4$ and $c_1 \approx 1.6 B_0$.

Exercise 18.3 The Earth's Bow Shock

[by Xinkai Wu/00]

(a) The momentum flux $\sim \rho v^2$ while the magnetic pressure generated by the earth's dipole field $\sim B^2/2\mu_0 \sim (B_E^2/2\mu_0)(r_E/r)^6$, noticing that for a dipole field $B \sim r^{-3}$. Balancing these two terms and plugging in the numbers: $B_E \sim 3 \times 10^{-5} T$, $r_E \sim 6 \times 10^6 m$ we get $r \sim 8.5 r_E \sim 5 \times 10^7 m$

(b) B&T eqn (18.23) gives: $E_1 + v_s B_1 = E_2 + v_s B_2$ and eqn (18.21) gives: $\rho_1(v_1 - v_s) = \rho_2(v_2 - v_s)$. In the infinite conductivity limit and applied to both sides of the shock front, eqn (18.5) gives: $E_1 = -v_1 B_1$ and $E_2 = -v_2 B_2$. Combining the above equations we get $B_2/B_1 = \rho_2/\rho_1 = (v_1 - v_s)/(v_2 - v_s)$. Namely the magnetic field strength will increase by the same ratio as the density on crossing the shock front. Intuitively we expect the compression to decrease as the field is increased, because increasing the field means increasing the magnetic pressure, which will in turn resist compression. To be more rigorous, let's look at a limiting case of equation (18.24): When B gets very large, the magnetic pressure term dominates and this equation gives $B_1 \approx B_2$, i.e. $\rho_1 \approx \rho_2$, which means there's almost no compression.

Exercise 18.6 Hartmann Flow

[by Guodong Wang/99]

The force balance equation is given by equation (18.34)

$$\nabla P = \mathbf{j} \times \mathbf{B} + \eta \nabla^2 \mathbf{v} \quad (44)$$

the x component of which is, using $\mathbf{B} = B_0 \mathbf{e}_z$ and $P = -Qx + p(z)$,

$$-Q = j_y B_0 + \eta \frac{d^2 v_x}{dz^2} \quad (45)$$

Now using equation (18.5) $\mathbf{j} = \kappa_e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ and $\mathbf{E} = E_0 \mathbf{e}_y$ we find

$$j_y = \kappa_e(E_0 - B_0 v_x) \quad (46)$$

Combining the above two equations, we get

$$\frac{d^2 v_x}{dz^2} - \frac{\kappa_e B_0^2}{\eta} v_x = -\frac{(Q + \kappa_e B_0 E_0)}{\eta} \quad (47)$$

(b) A special solution to this equation is

$$v_0 = \frac{Q + \kappa_e B_0 E_0}{\kappa_e B_0^2} \quad (48)$$

and the general solution is thus given by

$$v_x = v_0 + C_1 \cosh\left(\frac{Hz}{a}\right) + C_2 \sinh\left(\frac{Hz}{a}\right) \quad (49)$$

where $H = B_0 a \left(\frac{\kappa_e}{\eta}\right)^{1/2}$. Using the boundary condition $v_x(z = \pm a) = 0$ we find

$$C_1 = \frac{-v_0}{\cosh(H)}, \quad C_2 = 0 \quad (50)$$

and thus

$$v_x = \frac{Q + \kappa_e B_0 E_0}{\kappa_e B_0^2} \left[1 - \frac{\cosh(Hz/a)}{\cosh H} \right] \quad (51)$$