Physics 136a, Fall 2006: Solutions for Chapter 9

(compiled by Xinkai Wu, Kip Thorne, and Michael Cross)

A. Exercise 9.2 *Holographically reconstructed wave* [by Kip Thorne/99]

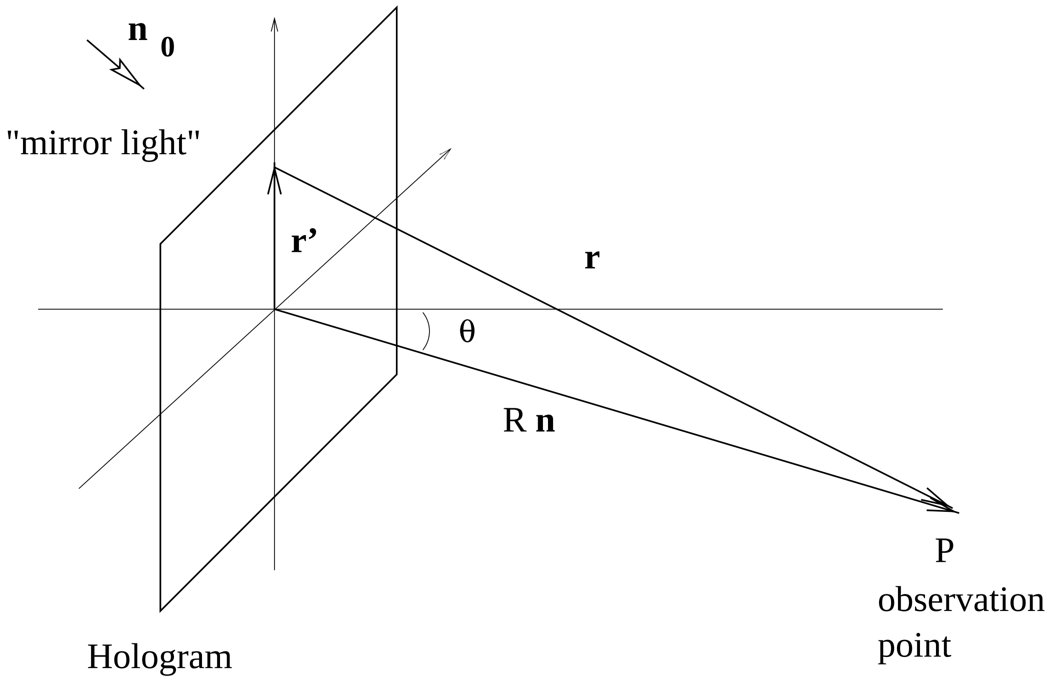

Figure 1: holographically reconstructed wave

(a) See Fig. 1, in which $r = |\mathbf{r}| = \left|R\mathbf{n} - \mathbf{r}'\right| \approx R - \mathbf{n}\cdot\mathbf{r}'$. Using the Helmholtz-Kirchhoff formula, we get

$$\psi_P = \frac{ik}{2\pi}\int d\mathbf{S}'\left(\frac{\mathbf{n}+\mathbf{n}_0}{2}\right)\frac{e^{i(kR-k\mathbf{r}'\cdot\mathbf{n})}}{r}\psi_0(x',y')$$

where

$$\begin{aligned}
\psi_0(x',y') &= T(x',y')e^{ik\mathbf{r}'\cdot\mathbf{n}_0} \\
&\sim \left(|O|^2 + M^2 + MOe^{-ik\mathbf{r}'\cdot\mathbf{n}_0} + MO^*e^{ik\mathbf{r}'\cdot\mathbf{n}_0}\right)e^{ik\mathbf{r}'\cdot\mathbf{n}_0} \\
&\sim (|O|^2 + M^2)e^{ik\mathbf{r}'\cdot\mathbf{n}_0} + MO + MO^*e^{2ik\mathbf{r}'\cdot\mathbf{n}_0}
\end{aligned}$$

and thus

$$\begin{aligned}
\psi_P &= \psi_P^{(1)} + \psi_P^{(2)} + \psi_P^{(3)} \\
\psi_P^{(1)} &= \frac{ik}{2\pi}\int d\mathbf{S}'\left(\frac{\mathbf{n}+\mathbf{n}_0}{2}\right)(|O|^2+M^2)\frac{e^{ikR}}{r}e^{ik\mathbf{r}'\cdot(\mathbf{n}_0-\mathbf{n})} \\
\psi_P^{(2)} &= \frac{ik}{2\pi}\int d\mathbf{S}'\left(\frac{\mathbf{n}+\mathbf{n}_0}{2}\right)MO\frac{e^{ikR}}{r}e^{-ik\mathbf{r}'\cdot\mathbf{n}} \\
\psi_P^{(3)} &= \frac{ik}{2\pi}\int d\mathbf{S}'\left(\frac{\mathbf{n}+\mathbf{n}_0}{2}\right)MO^*\frac{e^{ikR}}{r}e^{ik\mathbf{r}'\cdot(2\mathbf{n}_0-\mathbf{n})}
\end{aligned}$$

If the mirror wave had been absent and the photographic plate replaced by a window, then by using Helmholtz-Kirchhoff with $\tilde{\psi}_0 = O$ we would have gotten $\psi_P = \frac{ik}{2\pi}\int d\mathbf{S}'\left(\frac{\mathbf{n}+\mathbf{n}_0}{2}\right)O\frac{e^{ikR-ik\mathbf{r}'\cdot\mathbf{n}}}{r}$ which is the same as $\psi_P^{(2)}$ (to within a multiplicative constant).

The direction of propagation for different terms can be obtained by finding the stationary phase points for rapidly oscillating terms (note the last factor in the expressions for $\psi_P^{(1)}, \psi_P^{(2)}, \psi_P^{(3)}$ are rapidly oscillating, except in the direction of output wave). We find

for $\psi_P^{(1)}$, it's $\mathbf{n} = \mathbf{n}_0$, i.e. the wave propagates along the mirror wave direction.

for $\psi_P^{(2)}$, $\mathbf{n} = \mathbf{e}_z$, i.e. the wave propagates perpendicular to the hologram.

for $\psi_P^{(3)}$, $2sin\theta_0 = sin\theta$. (Note that if $\theta_0 < \pi/6$, then there exists a solution to this equation and the secondary image actually exists.)

(b) Now if all angles are small (paraxial optics), then

$$\psi_P^{(3)} = \frac{ik}{2\pi} \int d\mathbf{S}' M O^* \frac{e^{ikz}}{z} e^{ikr'(2\theta_0 - \theta)}, \quad \theta_0 \approx 0$$

The field at the object can be obtained by propagating back to point $(0, 0, -z)$

$$\psi_i = \frac{ik}{2\pi} \int d\mathbf{S}' O \frac{e^{-ikz}}{z} e^{ikr'\theta_i}, \quad \theta_s \approx \theta_i$$

i.e. $\psi_P(z) \propto \psi_i^*(-z)$, we see that indeed the secondary image resides in front of the hologram and is turned inside out. See Fig. 2.

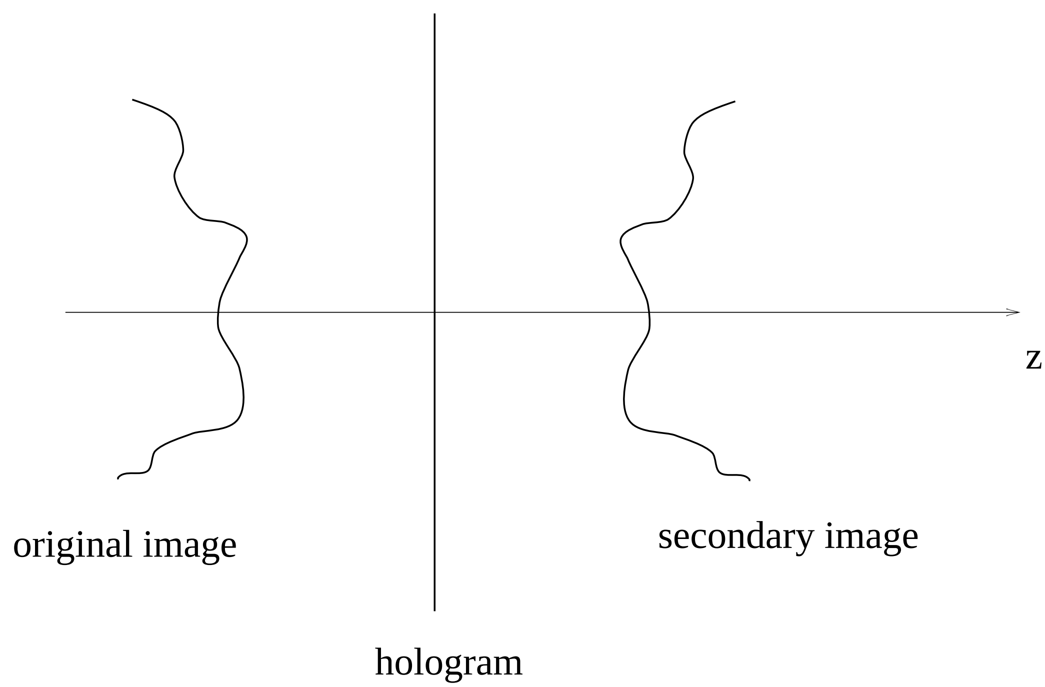

Figure 2: the secondary image

The angles in the wave scattered from the object and in the secondary wave are related by

$$-sin\theta_i = sin2\theta_0 - sin\theta_s$$

and for $\theta_i \approx 0$, if θ_0 is not small, then

$$\Delta\theta_i = cos\theta_s \Delta\theta_s \Rightarrow \Delta\theta_s = \frac{\Delta\theta_i}{cos2\theta_0} > \Delta\theta_i$$

i.e. the image is stretched.

(c) If light with wavenumber k is shined at the original angle θ_0 for reconstruction, then

$$\begin{aligned} \psi_0 &\sim MOe^{-ik_0 r' sin\theta_0} e^{ikr' sin\theta_0} \\ \psi_P^{(2)} &\sim \int d\mathbf{S}' \left(\frac{\mathbf{n}+\mathbf{n}_0}{2}\right) MO \frac{e^{ikR}}{r} e^{ir'(-k_0 sin\theta_0 + ksin\theta_0 - ksin\theta)} \end{aligned}$$

So again by looking for a stationary phase condition we see that the direction of propagation will be given by $\sin\theta = sin\theta_0 \left(1 - \frac{k_0}{k}\right)$. (the sign of θ is indicated in Fig. 3)

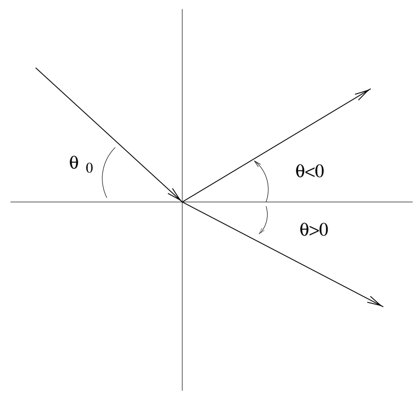

Figure 3: plane-parallel white light

$\theta_0 = \pi/4$, $k_0 = k_{green}$, and $sin\theta = \sin\theta_0 \left(1 - \frac{\lambda}{\lambda_{green}}\right)$.

(1) $\lambda = \lambda_{green}$, $sin\theta = 0, \theta = 0$

(2) $\lambda = \lambda_{red}$, $sin\theta = \frac{1}{\sqrt{2}} \left(1 - \frac{\lambda_{red}}{\lambda_{green}}\right) \approx \frac{1}{\sqrt{2}} \left(1 - \frac{700nm}{500nm}\right) \approx -0.28$. Thus $\theta \approx -16.4°$.

D. *Lorenz equations for lasers*

(a) The first equation is the equation for the driving of the electric field by the oscillating polarization. The second equation gives the stimulated emission, i.e., the driving of the polarization proportional to the product of the population inversion and the electric field. The last equation expresses the energy fed by transitions from the high to low energy levels into the electric field with a rate proportional to field×polarization. The dissipation terms (the various γ) are the field leaking from the cavity, damping of the oscillating dipole moments, e.g., by interatomic collisions, and decay from the upper to the lower level, e.g., by spontaneous emission.

After the substitution $Z \propto N_0 - N$ we see that the two sets of equations have the same structure, and so it is a question of scaling the variables and time to match coefficients. Comparing the "diagonal" term in the second equation we see that time is being measured in units of $\gamma_\perp^{-1}$. Introducing the scaled time $\tau = \gamma_\perp t$ gives

$$\frac{dA}{d\tau} = -\frac{\gamma}{\gamma_\perp}\left[A - \left(\frac{\omega}{2\varepsilon_0\gamma}\right)P\right] \tag{1a}$$

$$\frac{dP}{d\tau} = -P + \left(\frac{M^2}{\hbar\gamma_\perp}\right)NA \tag{1b}$$

$$\frac{dN}{dt} = -\frac{\gamma_\parallel}{\gamma_\perp}(N - N_0) - \left(\frac{1}{\hbar\gamma_\perp}\right)PA \tag{1c}$$

Now write $A = \lambda X$, $P = \mu Y$, $N_0 - N = -\nu Z$ to give

$$\frac{dX}{d\tau} = -\frac{\gamma}{\gamma_\perp}\left[X - \left(\frac{\mu}{\lambda}\frac{\omega}{2\varepsilon_0\gamma}\right)Y\right] \tag{2a}$$

$$\frac{dY}{d\tau} = -Y + \left(\frac{\lambda\nu}{\mu}\frac{M^2}{\hbar\gamma_\perp}\right)(\nu^{-1}N_0 + Z)X \tag{2b}$$

$$\frac{dZ}{dt} = -\frac{\gamma_\parallel}{\gamma_\perp}Z - \left(\frac{\lambda\mu}{\nu}\frac{1}{\hbar\gamma_\perp}\right)XY \tag{2c}$$

Comparing coefficients we find the Lorenz equations if

$$\frac{\mu}{\lambda} = \frac{2\varepsilon_0\gamma}{\omega} \tag{3a}$$

$$\frac{\lambda\nu}{\mu} = \frac{\hbar\gamma_\perp}{M^2} \tag{3b}$$

$$\frac{\lambda\mu}{\nu} = \hbar\gamma_\perp \tag{3c}$$

giving

$$\lambda = \frac{\hbar\gamma_\perp}{M},\ \mu = \frac{2\varepsilon_0\gamma M}{\omega\hbar\gamma_\perp},\ \nu = \frac{2\varepsilon_0\hbar\gamma\gamma_\perp}{\omega M^2}. \tag{4}$$

More important, we can evaluate the parameters of the Lorenz equations

$$r = \frac{\omega M^2 N_0}{2\varepsilon_0\hbar\gamma\gamma_\perp},\ \sigma = \frac{\gamma}{\gamma_\perp},\ b = \frac{\gamma_\parallel}{\gamma_\perp}. \tag{5}$$

The parameters σ, b are simply ratios of relaxation times. The parameter r can be expressed as

$$r = \frac{\omega\omega_d}{\gamma\gamma_\perp} \tag{6}$$

where $\hbar\omega_d = M^2N_0/2\varepsilon_0$ is roughly the energy of two atomic atomic dipole moments at a separation corresponding to the density N_0.

Solved by J Dort

Exercise 9.8 : Efficiency of Frequency Doubling

$$\frac{dE^{(2\omega)}}{dz} \simeq -2i\,\frac{k_3}{n_3^2}\,\chi\,E^2 , \quad \omega_3 = 2\omega , \quad k_3 = 2k , \; n_3 = n$$

$$\Rightarrow E^{(2\omega)} \simeq -4\,\frac{ik}{n^2}\,\chi\,E^2 \cdot l , \quad P = \underset{\text{area}}{A} \cdot \frac{1}{2}\,\frac{c}{n}\,\varepsilon_0\,\left|E^{(2\omega)}\right|^2$$

This is the crudest possible approximation, since the amplitude in the original frequency channel is treated as constant. If the power in the doubled frequency channel is almost the same:

$$\Rightarrow |E| \simeq \frac{n^2}{4k\chi l} = \left(\frac{2nP}{Ac\varepsilon_0}\right)^{1/2} \simeq \frac{1}{d_0}\left(\frac{2nP}{c\varepsilon_0}\right)^{1/2}$$

$$\Rightarrow d_0 \simeq \frac{4k\chi}{n^2}\left(\frac{2nP}{\varepsilon_0 c}\right)^{1/2} \cdot l$$

d_0: diameter of focused light

l: length of system.

Plug in the numbers $d_0 \simeq 1$ mm

From the discussion in 7-26, if the beam is in the Fresnel region the beam size is nearly constant. This is satisfied if $\sqrt{\lambda l} \ll d$. Here $\lambda \sim 10^{-6}$ m, $d_0 \sim 1$ mm. Hence if $l \ll 1$ m then the diffraction effect is small and this condition is satisfied for the crysta

Solved by A. Dvoretskii

9.7 Growth equation in realistic wave-wave mixing

Starting from Maxwell's equation one can obtain (see solution for 9.5)

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} + \mu_0\frac{\partial^2}{\partial t^2}\vec{P}^L = -\mu_0\frac{\partial^2}{\partial t^2}\vec{P}^{NL}$$

where $\vec{P}^L$ is the part of the polarization vector linear in E.

or $\hat{L}_{ij}\cdot E_j = -\mu_0\frac{\partial^2}{\partial t^2}P_i^{NL}$

where $\det|L_{ij}|=0$ gives the dispersion relation for linear wave propagation.

Now suppose $E_j = \mathcal{E}_j(\vec{r},t)\,e^{i\vec{k}\vec{r}-i\omega t}$ and the envelope $\mathcal{E}(\vec{r},t)$ varies slowly in time and space. I.e $Kl\gg1$, $\omega\tau\gg1$ where l,τ are the length and time characteristic variations. Then

$$L_{ij}E_j = \underbrace{\mathcal{E}_j\cdot L_{ij}(K,\omega)}_{=0} + \frac{\partial L_{ij}(K,\omega)}{\partial(-i\omega)}\frac{\partial\mathcal{E}_j}{\partial t} + \frac{\partial L_{ij}(K,\omega)}{\partial(ik_m)}\frac{\partial\mathcal{E}_j}{\partial x_m} = -\mu_0\frac{\partial^2}{\partial t^2}P_i^{NL}$$

because of the wave equation

other terms (e.g. $\frac{\partial^2 L_{ij}(K,\omega)}{\partial^2(-i\omega)}\frac{\partial^2\mathcal{E}_j}{\partial t^2}$) are suppressed by $1/\omega\tau$ or $1/Kl$

$$\frac{\partial L_{ij}}{\partial\omega}\frac{\partial\mathcal{E}_j}{\partial t} - \frac{\partial L_{ij}}{\partial\vec{K}}\cdot\frac{\partial\mathcal{E}_j}{\partial\vec{x}} = i\mu_0\frac{\partial^2}{\partial t^2}P_i^{NL}$$

(b) The wave for which the eigenvalue $\lambda_1=0$ satisfies the dispersion relation and propagates through the medium.

$$\frac{\partial\lambda_1}{\partial\omega}\frac{\partial\mathcal{E}_i^{(1)}}{\partial t} - \frac{\partial\lambda_1}{\partial k_j}\frac{\partial\mathcal{E}_i^{(1)}}{\partial x_j} = \frac{\partial\lambda_1}{\partial\omega}\left(\frac{\partial\mathcal{E}_i^{(1)}}{\partial t} - \left(\frac{\partial\lambda_1}{\partial k_j}\Big/\frac{\partial\lambda_1}{\partial\omega}\right)\frac{\partial\mathcal{E}_i^{(1)}}{\partial x_j}\right) = i\mu_0\frac{\partial^2}{\partial t^2}\mathcal{P}_i^{NL}$$

Since $\mathcal{E}$ is the envelope we see that the group velocity is $\vec{V}_g^{(1)} = -\frac{\partial\lambda/\partial\vec{k}}{\partial\lambda_1/\partial\omega}$

c) $\frac{\partial \mathcal{E}_i^{(1)}}{\partial t} + \vec{V}_g^{(1)} \frac{\partial \mathcal{E}_i^{(1)}}{\partial \vec{x}} = \frac{d}{dt}\mathcal{E}_i = \frac{i\mu_0 \frac{\partial^2}{\partial t^2} P_i^{NL}}{\partial \lambda_i / \partial \omega} = \frac{-i\mu_0 \omega^2 P_i^{NL}}{\partial \lambda_i / \partial \omega}$

(d). For 3-wave mixing $\omega = \omega_1 + \omega_2$

$P_i^{NL} = \varepsilon_0 \chi_{ijk} \mathcal{E}_j^{(1)} \mathcal{E}_k^{(2)}$, so

$$\frac{d\mathcal{E}_i}{dt} = -i \frac{\omega^2/c^2}{\partial \lambda_i / \partial \omega} \chi_{ijk} \mathcal{E}_j^{(1)} \mathcal{E}_k^{(2)}$$

$$\frac{d\mathcal{E}_i}{ds} = -i \frac{\omega^2/c^2}{|V_g| \partial \lambda_i / \partial \omega} \chi_{ijk} \mathcal{E}_j^{(1)} \mathcal{E}_k^{(2)} = -2i\alpha \frac{K_3}{n_3^2} \chi_{ijk} \mathcal{E}_j^{(1)} \mathcal{E}_k^{(2)}$$

$$\alpha = \frac{K}{2} \frac{1}{|\partial \lambda / \partial K|}$$