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Landau-Lifshitz formulation of Einstein Field Equations

- In any coordinate system define:

$$H^{\mu\alpha\nu\beta}_{\text{L-L}} = \mathfrak{g}^{\mu\nu}\mathfrak{g}^{\alpha\beta} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\mu\beta}, \tag{20.20}$$

where $\mathfrak{g}^{\mu\nu} \equiv (-g)^{1/2} g^{\mu\nu}$. Landau and Lifshitz show that Einstein's equations can be written in the form

$$H^{\mu\alpha\nu\beta}_{\text{L-L},\alpha\beta} = 16\pi(-g)(T^{\mu\nu} + t^{\mu\nu}_{\text{L-L}}), \tag{20.21}$$

where the Landau-Lifshitz pseudotensor components

$$\begin{aligned}(-g)\, t^{\alpha\beta}_{\text{L-L}} = \frac{1}{16\pi}\Big\{ & \mathfrak{g}^{\alpha\beta}{}_{,\lambda}\mathfrak{g}^{\lambda\mu}{}_{,\mu} - \mathfrak{g}^{\alpha\lambda}{}_{,\lambda}\mathfrak{g}^{\beta\mu}{}_{,\mu} + \frac{1}{2} g^{\alpha\beta} g_{\lambda\mu}\mathfrak{g}^{\lambda\nu}{}_{,\rho}\mathfrak{g}^{\rho\mu}{}_{,\nu} \\ & - (g^{\alpha\lambda} g_{\mu\nu}\mathfrak{g}^{\beta\nu}{}_{,\rho}\mathfrak{g}^{\mu\rho}{}_{,\lambda} + g^{\beta\lambda} g_{\mu\nu}\mathfrak{g}^{\alpha\nu}{}_{,\rho}\mathfrak{g}^{\mu\rho}{}_{,\lambda}) + g_{\lambda\mu} g^{\nu\rho}\mathfrak{g}^{\alpha\lambda}{}_{,\nu}\mathfrak{g}^{\beta\mu}{}_{,\rho} \\ & + \frac{1}{8}(2g^{\alpha\lambda}g^{\beta\mu} - g^{\alpha\beta}g^{\lambda\mu})(2g_{\nu\rho}g_{\sigma\tau} - g_{\rho\sigma}g_{\nu\tau})\mathfrak{g}^{\nu\tau}{}_{,\lambda}\mathfrak{g}^{\rho\sigma}{}_{,\mu}\Big\}\end{aligned} \tag{20.22}$$

are precisely quadratic in the first derivatives of the metric. Moreover, the equations of motion $T^{\alpha\beta}{}_{;\beta} = 0$ take the form

$$\left[(-g)(T^{\mu\nu} + t^{\mu\nu}_{\text{L-L}})\right]_{,\nu} = 0 \tag{A}$$

↑ comma (partial derivative)

In these equations

$$\mathfrak{g}^{\alpha\beta} \equiv \sqrt{-g}\; g^{\alpha\beta} \tag{B}$$

↑ components of metric

In the weak-field (linearized gravity) limit

$$\mathfrak{g}^{\alpha\beta} = \eta^{\alpha\beta} - \bar{h}^{\alpha\beta}. \tag{C}$$

■ If we impose "deDonder gauge" (also called "Harmonic gauge" or "Harmonic coordinates" — the nonlinear generalization of Lorentz gauge)

$$\mathfrak{g}^{\alpha\beta}{}_{,\beta} = 0 \qquad \text{(D)}$$

Then the field equations (20.21), (20.22) simplify to

$$\underbrace{\mathfrak{g}^{\alpha\beta}\,\mathfrak{g}^{\mu\nu}{}_{,\alpha\beta}} = 16\pi(-g)\left(T^{\mu\nu} + t^{\mu\nu}_{L-L}\right) + \mathfrak{g}^{\alpha\nu}{}_{,\beta}\,\mathfrak{g}^{\mu\beta}{}_{,\alpha} \qquad \text{(E)}$$

and the first two terms of $t^{\mu\nu}_{L-L}$ vanish

$$= \sqrt{-g}\; g^{\alpha\beta}\,\mathfrak{g}^{\mu\nu}{}_{;\alpha\beta}$$

= Curved-spacetime wave operator acting on $\mathfrak{g}^{\alpha\beta}$

— which demonstrates that the Einstein equations are fundamentally hyperbolic with characteristics that are null rays.

■ In situations where we can introduce nearly Lorentz coordinates, it is useful to define $\bar{h}^{\alpha\beta}$ by [cf. Eq. (C)]

$$\mathfrak{g}^{\alpha\beta} \equiv \eta^{\alpha\beta} - \bar{h}^{\alpha\beta} \qquad \text{(exact definition; no linearization)} \qquad \text{(F)}$$

Then the gauge conditions (D) are

$$\bar{h}^{\alpha\beta}{}_{,\beta} = 0 \qquad \text{(exact; no linearization)} \qquad \text{(G)}$$

and the field equations (E) become the following nonlinear wave equation in flat spacetime

$$\eta^{\alpha\beta}\,\bar{h}^{\mu\nu}{}_{,\alpha\beta} = -16\pi\,\overbrace{(-g)\left(T^{\mu\nu}+t^{\mu\nu}_{L-L}\right)}^{\text{can be expanded as a power series in } \bar{h}^{\alpha\beta}.}$$

$$-\bar{h}^{\alpha\nu}{}_{,\beta}\,\bar{h}^{\mu\beta}{}_{,\alpha} + \underbrace{\bar{h}^{\alpha\beta}\,\bar{h}^{\mu\nu}{}_{,\alpha\beta}} \qquad (H)$$

because this involves second derivatives, it is really modifying the characteristics to make them rays of curved spacetime rather than flat

where $t^{\mu\nu}_{L-L}$ has the form (20.22) with each $\mathfrak{g}^{\alpha\beta}{}_{,\lambda}$ replaced by $-\bar{h}^{\alpha\beta}{}_{,\lambda}$.

The field equations (H) can be solved iteratively accurate to linear order, then quadratic order, then cubic order, ... in $\bar{h}^{\alpha\beta}$.