Ph 236a: General Relativity — 3 October 2007
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WEEK 1: FOUNDATIONS OF SPECIAL RELATIVITY

Recommended Reading related to my first week lectures: This looks like a lot of reading. However, most students will already be familiar with a large portion of this material. For those students, I urge that you try to understand the material from the geometric point of view that I am developing, which may be new to you. That viewpoint is central to general relativity.

1. Blandford and Thorne, *Applications of Classical Physics*, Sections 1.1–1.4 (pages 1–20) and Section 1.7.3 (pages 34–36) of Chapter 1, version 0601.2.K. Note: This book is not yet published. It is on line at http://www.pma.caltech.edu/Courses/ph136/yr2006/text.html
2. MTW (Misner, Thorne and Wheeler, *Gravitation*): Chapter 1. A pdf file of this chapter will be available sometime Thursday in the password-protected part of our website. Because of copyright law, please do not give this file to anyone except people in our class.
3. MTW, Section 2.1–2.6. A pdf file of this chapter will be available in the password-protected part of our website sometime Thursday. Because of copyright law, please do not give this file to anyone except people in our class.

Possible Supplementary Reading:

4. Bernard F. Schutz (*A First Course in General Relativity*): Chapter 1.
5. Robert M. Wald (*General Relativity*): Chapter 1.
6. Sean Caroll (*An Introduction to Spacetime and Geometry*): Sections 1.1–1.3.
7. James B. Hartle (*Gravity*): Chapters 1–4. This is especially good and detailed if you have little or no previous exposure to special relativity.

Problems

Each problem is worth 10 points. The maximum number of points that will be given for this problem set is 50 points. The number of points you will get will be the maximum of 50, and the sum of the points you actually achieve on all the problems you attempt. Please work those problems that are most useful for you, given your previous preparation and experience. If you have never done much with spacetime diagrams, please give special attention to problems 4 and 5. See the document *Course procedures* on our class website for a discussion of homework and grading in this course.

1. **Geometrized Units.**

   "Geometrized units" are units in which the speed of light

   $$c \equiv 2.99792458000000000000000000000000000000000000\text{cm/sec}$$

   is set to unity, and thereby we regard one second as equal to 2.99792458 centimeters. (Later we will also set Newton's gravitation constant to unity.)

a. How can experimental physicists have actually measured the speed of light to the enormous accuracy indicated by the above string of digits, or did they? What are the next 17 digits in the above value of c? How do you know?

b. What is one year, expressed in centimeters?

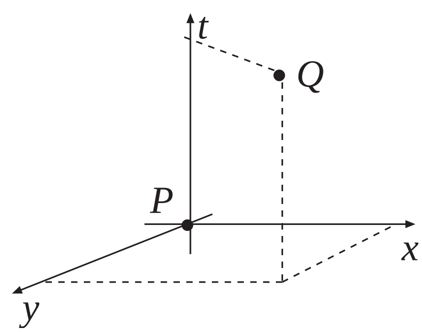
c. What is your age, in centimeters?

d. What is your height, in seconds?

e. The following equations are written in geometrized units. Restore the factors of c so they are in cgs units. [This can be done by simply inserting whatever factors of c are required to make the equations dimensionally consistent, in cgs units.]
(Planck length)$= \sqrt{G\hbar}$, where $\hbar$ is Planck's constant and G is Newton's gravitation constant.
(Energy density in an electromagnetic wave)$= (\mathbf{E}^2 + \mathbf{B}^2)/(8\pi)$.
(Lorentz force on a proton)$= e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$.

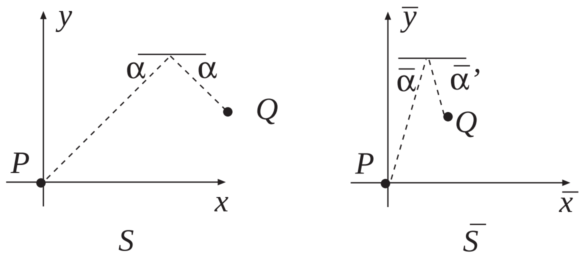
2. **Proof of Invariance of a Timelike Interval.**
Note: The solution is sketched briefly in Sec. 1.2.2 and Exercise 1.2 of Blandford and Thorne, and a very leisurely version of the solution is given in Secs. 3.6 and 3.7 of Taylor and Wheeler, *Spacetime Physics* (1992).

Consider two events $\mathcal{P}$ and $\mathcal{Q}$ in spacetime, with a timelike separation vector $\vec{A}$. Examine these events in two different reference frames $\mathcal{S}$ and $\bar{\mathcal{S}}$ which move with speed v relative to each other. Choose the origins of the two frames' spacetime coordinates to coincide, and to be at the event $\mathcal{P}$, and orient the spatial axes of the two frames so their relative motion is in the x direction and $\mathcal{Q}$ lies in the $x - y$ plane. The following diagram depicts this in a spacetime diagram drawn from the viewpoint of frame $\mathcal{S}$.



a. Convince yourself that, wherever may be the events $\mathcal{P}$ and $\mathcal{Q}$, the origins and axes of the two frames can be adjusted as described above.

The following experiment is a foundation for proving the invariance of the interval between $\mathcal{P}$ and $\mathcal{Q}$. The experiment is sketched below in two purely spatial diagrams (time not shown) from the frames' two different viewpoints.



A photon (light pulse) is emitted from $\mathcal{P}$, and travels along a straight line in the x-y plane until it hits a mirror that reflects it; the photon then travels again in a straight line in the x-y plane, arriving at the event $\mathcal{Q}$. The position of the reflecting mirror is adjusted so the photon reaches the spatial location of $\mathcal{Q}$ precisely at the time of $\mathcal{Q}$. The state of motion of the mirror is not important; the key thing is that, as seen in frame $\mathcal{S}$, the photon's direction of motion makes an angle α with the x axis that is the same before and after reflection ("angle of incidence equals angle of reflection"), as shown in the diagram. In the following do *not* use the Lorentz transformation equations. Assume that they have not yet been derived; we are working our way toward them, and our first step is to derive the invariance of the interval from the Principle of Relativity.

b. Use the Principle of Relativity to show that the heights of the reflection point are the same in the two frames, $y_{\rm refl} = \bar{y}_{\rm refl}$, and that the angles of incidence and reflection are equal in frame $\bar{\mathcal{S}}$, $\bar{\alpha} = \bar{\alpha}'$, just as they are equal in frame $\mathcal{S}$.

c. Use the Principle of Relativity, the constancy of the speed of light, and simple geometric considerations to show that the interval between $\mathcal{P}$ and $\mathcal{Q}$ is the same, as computed in the two reference frames:

$$-(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = -(\Delta \bar{t})^2 + (\Delta \bar{x})^2 + (\Delta \bar{y})^2 + (\Delta \bar{z})^2 \ .$$

3. **Lorentz Transformation**

For the two frames of Exercise 2, assume that the relationship between the coordinates is linear.

a. Use arguments from the Principle of Relativity and symmetry arguments to show that for any event (e.g. $\mathcal{Q}$), $y = \bar{y}$ and similarly $z = \bar{z}$; and thus it is only x and t that get mixed up among each other.

b. Write the transformation for x and t as

$$x = A\bar{x} + B\bar{t} \ , \quad t = C\bar{t} + D\bar{x} \ .$$

Use the invariance of the interval, and the fact that frame $\bar{\mathcal{S}}$ moves at speed v in the x direction as seen from frame $\mathcal{S}$, to show that A, B, C, and D have the standard values for a Lorentz boost in the x direction, $A = C = \gamma$, $B = D = v\gamma$, where $\gamma = 1/\sqrt{1 - v^2}$.

4. **Spacetime Diagram – The most important exercise of this problem set — unless it is already fully familiar to you, in which case skip it!**

For the two reference frames of Exercises 2 and 3, draw a spacetime diagram for the $\bar{x}$-$\bar{t}$ "plane" from the viewpoint of frame $\mathcal{S}$. On this diagram, draw the x and t axes of frame $\bar{\mathcal{S}}$. Use this diagram to prove the following results:

a. Events that are simultaneous as seen in frame $\mathcal{S}$ are not simultaneous as seen in $\bar{\mathcal{S}}$. Which events occur first and which later?
b. An ideal clock at rest in frame $\mathcal{S}$ appears, as seen from $\bar{\mathcal{S}}$, to tick abnormally slowly; and conversely, an ideal clock at rest in frame $\bar{\mathcal{S}}$ appears, as seen from $\mathcal{S}$, to tick abnormally slowly. Use the geometry of the diagram to show that the slow-down factor is $\sqrt{1-v^2}$.
c. An ideal rod at rest in frame $\mathcal{S}$ appears, as seen from $\bar{\mathcal{S}}$, to be contracted by a factor $\sqrt{1-v^2}$; and conversely, an ideal rod at rest in frame $\bar{\mathcal{S}}$ appears, as seen from $\mathcal{S}$, to be contracted by that same factor.
d. Draw a spacetime diagram from the viewpoint of the frame $\bar{\mathcal{S}}$, and repeat the proof of the above three effects.

5. **Another Spacetime Diagram** [adapted from Schutz]
An experimeter in an inertial frame $\mathcal{S}$ performs the following experiment: Two bursts of particles of speed $v = 0.5$ are emitted from $x = 0$ and $t = -200$cm, one traveling in the $+x$ direction and the other in the $-x$ direction. These particles encounter detectors located at $x = \pm 200$cm. After a delay of 50cm time, the detectors send signals back to $x = 0$ at speed $v = 0.75$.
a. Draw a spacetime diagram of this experiment using the coordinates of $\mathcal{S}$.
b. The signals arrive back at $x = 0$ at the same event. (Make sure your spacetime diagram shows this.) From this, the experimenter concludes that the particle detectors did indeed send out their signals simultaneously, since the experimenter knows they are equal distances from $x = 0$. An observer resides in an inertial frame $\bar{\mathcal{S}}$ moving at speed $v = 0.75$ in the $+x$ direction relative to $\mathcal{S}$. Draw this frame's axes on your frame-$\mathcal{S}$ spacetime diagram. Then draw a spacetime diagram of the above experiment from this new frame's viewpoint, i.e. using the coordinates of $\bar{\mathcal{S}}$. Use this diagram to deduce whether the particle detectors, as seen by this observer, send out their signals simultaneously; and if not simultaneous, which one sends its signal first and how much sooner.
c. Compute the interval Δs^2 between the events at which the particle detectors emit their signals, using both the coordinates of $\mathcal{S}$ and of $\bar{\mathcal{S}}$.

6. **Frame-independent Force Law for a Scalar Field**
Note: This is an exercise to give you experience thinking about frame-independent laws of physics.
a. Convince yourself that for any vector $\vec{A}$ at some event $\mathcal{P}$ there exists a curve $\mathcal{P}(\lambda)$ to which it is tangent, $\vec{A} = d\mathcal{P}/d\lambda$; i.e., any vector can be regarded as tangent to some curve. Convince yourself that there is actually an infinite number of curves to which $\vec{A}$ (in the tangent space at $\mathcal{P}$) is tangent.
b. We define the *gradient* or *exterior derivative* of a scalar field Φ to be the one-form (linear function of vectors) $\mathbf{d\Phi}$ whose value on $\vec{A} = d\mathcal{P}/d\lambda$ is

$$\mathbf{d}\Phi\left(\frac{d\mathcal{P}}{d\lambda}\right) = \frac{d}{d\lambda}\Phi[\mathcal{P}(\zeta)]\ .$$

Show that this definition of $\mathbf{d}\Phi$ really is linear in the vector $\vec{A} = d\mathcal{P}/d\lambda$.

c. Suppose that nature is endowed with a scalar field $\Phi(\mathcal{P})$, to which a particle with scalar charge q can couple, thereby producing a force that is linear in q and linear in the gradient (exterior derivative) of Φ. Write down the simplest force law you can think of that has these properties.

d. Does your force law preserve the rest mass of the particle? If not, then find a generalization that *does* preserve the particle's rest mass. Note: You may want to try inserting some dependence on the particle's 4-velocity $\vec{u}$ into the force law.

e. In the momentary rest frame of the particle, what form does your new, rest-mass-conserving force law take? i.e., what are its time component and its spatial components?