Ph 236a
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WEEK 10: Geometric Optics, GW Stress-Energy Tensor, and General Relativity as a Nonlinear Field Theory in Flat Spacetime
REVISION 1 - ON DECEMBER 6

NOTE: Revisions include:

A. Ref. 2.d has been expanded to include additional sections in my book.

B. Problem 8: The wave equations you are asked to derive are valid only in vaccum (i.e. when the stress-energy tensor vanishes, i.e. when the Ricci tensor vanishes), so the phrase "in vacuum" has been added, twice.

C. Problem 9: I am working on this but have not yet finished. I hope to give it to you tomorrow, Friday, in a second revision of the problem set. - Kip

Recommended Reading:

1. **Stress-energy tensor for gravitational waves**
   a. For a quick derivation, omitting lots of details, along the lines of what I did in class, read Sec. 26.3.4 of Chapter 26 of Blandford and Thorne, version 0626.1.K.pdf; available at http://www.pma.caltech.edu/Courses/ph136/yr2006/ .
   b. For a more detailed treatment, read Sections 35.13–35.15 of MTW.
   c. For the energy and angular momentum carried away from a slow-motion source by gravitational waves, computed as a surface integral of the GW energy flux $T^{0r}_{\rm GW}$, read page 992 of Sec. 36.7 of MTW.
2. **Geometric optics for wave propagation in curved spacetime.**
   a. I suggest beginning with Section 22.5 of MTW. This is a leisurely treatment of geometric optics for quasi-monochromatic electromagnetic waves, giving a lot more detail than I did in my lectures.
   b. Then read (but do not do) MTW Exercise 35.15, which is the analogous geometric-optics treatment of quasi-monochromatic gravitational waves.
   c. Then read Sections 26.3.5 and 26.3.6 of Blandford and Thorne, Chapter 26, version 0626.1.K.pdf . This writes down the geometric optics solution in the time domain in the form I gave in class, and verifies that it satisfies the Einstein field equations.
   d. Finally read Sections 5.A,B,D 5.D of an unpublished 1989 book that I have written on gravitational waves (on our website), which presents and derives the geometric optics equations in the manner that I did in class.
3. **General relativity as a nonlinear field theory in flat spacetime, and its use to derive the evolution laws for the mass, momentum, and angular momentum of a semi-isolated body.**
   a. I suggest beginning by reading MTW chapter 19 ("Mass and Angular Momentum of a Gravitating System") and Secs. 20.1–20.5 (nonlinear field theory and derivation of evolution laws).
   b. Having read MTW, I suggest reading pages 341–344 of Sec. 100 ("The energy-momentum pseudotensor") of Landau and Lifschitz, *The Classical Theory of Fields*

(revised second edition, 1962; in a later edition this might be section 101 and different page numbers). This sketches a derivation of the nonliner field theory equations that I presented in class and that are discussed in MTW — though in a somewhat different notation than I used in class and than MTW.

c. The nonlinear field theory equations and evolution equations are written in a handout that I passed out in class, titled "Landau-Lifshitz Formulation of the Einstein Field Equations". That handout is on our website immediately after this Assignment 10.

Supplementary Reading on GW Stress-Energy Tensor

4. For computation of the energy, linear momentum, and angular momentum carried by gravitational waves, as integrals over a sphere surrounding the source, with the answers expressed as sums over the source's multipole moments, see Thorne, *Reviews of Modern Physics* , **52**, 299 (1980), Sections 4B,C,D (pages 317–319).
5. For a beautiful computation of the energy flux in a gravitational wave based on an analysis of the energy extracted from the wave by a dense collection of mechanical oscillators, see Sec. 9.4 of Schutz, *A First Course in General Relativity*

Supplementary Reading on Geometric Optics:

6. Sections 6.2 and 6.3 of Version 0406.3.K.pdf of Chapter 6 of Blandford and Thorne, available at http://www.pma.caltech.edu/Courses/ph136/yr2004/ . This treats geometric optics for most any type of wave (electromagnetic waves in dielectric media, sound waves in solids such as the interior of the earth, sound waves in fluids, ...).

Supplementary Reading on Applications of the Nonlinear Field Theory Formulation of General Relativity:

7. MTW Sec. 20.6 on "Equations of Motion Derived from Field Equations". This material [due to John Wheeler] describes the conceptual foundations for the derivation of the laws of motion and precession of a self-graviting body (e.g., a black hole) moving through curved spacetime; and it uses those foundations to show that [if one ignores coupling of the body's multipole moments to the external spacetime curvature], the body moves along a geodesic.
8. Kip S. Thorne and James B. Hartle, "Laws of Motion and Precession for Black Holes and Other Bodies", *Physical Review D*, **31**, 1815 (1985). This paper combines the Landau-Lifshitz formalism (general relativity as a field theory in flat spacetime; Refs. 3.b and 3.c above) with the conceptual foundations for laws of motion given in MTW (Ref. 5 above), to derive the actual laws and equations of motion and precession for compact bodies such as neutron stars and black holes.
9. Kip S. Thorne and Yekta Gürsel, "The Free Precession of Slowly Rotating Neutron Stars: Rigid-Body Motion in General Relativity", *Monthly Notices of the Royal Astronomical Society*, **205**, 809 (1983). This paper uses the Landau-Lifshitz formalism to prove that the equations of free precession for a slowly and rigidly rotating body are independent of the strength of the body's internal gravity.

Problems [Each problem is worth 10 points unless otherwise indicated. The maximum number of points you can get from this set is 50 points plus whatever you earn for problem 9.]

1. **Stress-Energy Tensor for a Plane Gravitational Wave in terms of h_+ and $h_\times$**

   a. In MTW it is shown that, in any Lorenz gauge ($\bar{h}^{\alpha\beta}_{\ \ |\beta} = 0$) in which $\bar{h}^{\alpha\beta}$ has been made traceless by a further gauge specialization (e.g., in TT gauge), the wave's stress-energy tensor has the form

$$T^{\rm GW}_{\mu\nu} = \frac{1}{32\pi} \langle \bar{h}_{\alpha\beta|\mu} \bar{h}^{\alpha\beta}{}_{|\nu} \rangle \ . \qquad (6)$$

   a. Consider a plane gravitational wave propagating in the z direction as seen in some local Lorentz reference frame. Show that the wave's stress-energy tensor in this case has as its only nonzero components

$$T^{00}_{\rm GW} = T^{0z}_{\rm GW} = T^{z0}_{\rm GW} = T^{zz}_{\rm GW} = \frac{1}{16\pi} \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle \ , \qquad (7)$$

   where the dots mean $\partial/\partial t$.

   b. Show that, in the language of problem 6 of Assignment 8, these waves' energy density and flux have boost weight two (i.e. under a boost along the propagation direction, their magnitudes are changed by the square of the doppler shift factor). Deduce this from the vanishing boost weight of h_+ and $h_\times$ (problem 7.b of Assignment 8, with a typo corrected: the words "spin weight" should be "boost weight"). Also deduce it from the Lorentz transformation law for any stress-energy tensor.

2. **Cross Sectional Area of a Bundle of Rays** [15 Points]

   a. MTW Exercise 22.13.

   b. Show that the square root of the cross sectional area $\mathcal{A}$ of a bundle of rays is propagated along the bundle's central ray $\mathcal{C}_0$ according to the equation $\nabla_{\vec{k}} \sqrt{\mathcal{A}} = \frac{1}{2}(\vec{\nabla} \cdot \vec{k})\sqrt{\mathcal{A}}$. This is the same propagation law as for the quantity r that appears in the denominator of the geometric-optics solution $h_{+,\times} = Q_{+,\times}(\tau; \theta, \phi)/r$ that I discussed in my lectures this week and appears in Eq. (26.80) of Blandford and Thorne. Therefore, r is proportional to the square-root of the cross-sectional area of a bundle of rays, which means that the waves' energy flux (which is proportional to $\langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle$), dies out as $1/r^2 \propto 1/\mathcal{A}$, as one would expect is required for energy conservation.

3. **Gravitons**

   Read Exercise 35.15 of MTW. Solve Exercise 35.16 of MTW.

4. **Gravitational Waves from a Binary Star at the Center of a Spherical Galaxy** [15 Points]

   A binary star system is at the center of a spherical galaxy, whose background spacetime metric has the standard static, spherical form (same as that for the interior of a spherical star)

$$ds^2 = -e^{2\Phi(r)} dt^2 + e^{2\Lambda(r)} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \ , \qquad (1)$$

   with $\Lambda = 0$, $\Phi = \Phi_c$ at the galaxy's center $r = 0$, and $\Lambda \to 0$, $\Phi \to 0$ at radial infinity. Describe the waves' propagation through the galaxy by geometric optics.

a. Because the local wave zone is very small compared to the galaxy and compared to the radius of curvature of spacetime produced by the galaxy's gravity, the metric can be regarded as flat there:

$$ds^2 = -d\hat{t}^2 + d\hat{r}^2 + \hat{r}^2(d\theta^2 + \sin^2\theta d\phi^2) \; . \tag{2}$$

(Here I neglect the galaxy's tidal fields and the binary's mass and angular momentum.) The angular coordinates θ and ϕ are the same as those of the galaxy's coordinate system, Eq. (1). What is the relationship between $(\hat{t}, \hat{r})$ and (t, r)?

b. In the local wave zone, introduce gravitational-wave polarization tensors

$$\mathbf{e}_+ = \vec{e}_{\hat{\theta}} \otimes \vec{e}_{\hat{\theta}} - \vec{e}_{\hat{\phi}} \otimes \vec{e}_{\hat{\phi}} \; , \quad \mathbf{e}_\times = \vec{e}_{\hat{\theta}} \otimes \vec{e}_{\hat{\phi}} + \vec{e}_{\hat{\phi}} \otimes \vec{e}_{\hat{\theta}} \; , \tag{3}$$

where $\vec{e}_{\hat{\theta}}$ and $\vec{e}_{\hat{\phi}}$ are unit basis vectors pointing in the θ and ϕ directions. The orientation of the spherical coordinates is chosen so the binary's orbital plane is $\theta = \pi/2$, i.e. it rotates around the $\theta = 0$ axis. The resulting gravitational waves, in the binary's LARF and with the above choice of polarization tensors, have the following waveforms (which you derived last week for an equal-mass binary, but in a slightly different notation):

$$\begin{aligned} h_+ &= 2(1 + \cos^2\theta)\frac{\mu}{\hat{r}}(\pi M f)^{2/3}\cos(2\pi f t) \; , \\ h_\times &= 4\cos\theta\frac{\mu}{\hat{r}}(\pi M f)^{2/3}\sin(2\pi f t) \; . \end{aligned} \tag{4}$$

Here M is the binary's total mass, μ is its reduced mass, and $f = (1/\pi)\sqrt{M/a^3}$ (with a the separation between its stars) is its gravitational-wave frequency. Show that, as seen on the binary's rotation axis, these waves are circularly polarized, while as seen in its orbital plane, they are linearly polarized.

c. These waves propagate out through the galaxy. What are the rays on which they propagate? What are the parallel propagated polarization tensors (you should be able to deduce them from simple symmetry considerations). What are the waveforms measured by an observer far outside the galaxy, who is at rest with respect to the galaxy and the binary?

d. Consider an observer on the binary's rotation axis and far outside the galaxy. This observer is moving toward the galaxy with very high speed v. What are the waveforms measured by this observer?

5. 4-Momentum Flux Integral Applied to Schwarzschild
The flux integral for 4-momentum $P^\mu = \frac{1}{16\pi}\int_S H^{\mu\alpha 0 j}{}_{,\alpha} d^2 S_j$ [MTW Eq. (20.9)] is valid in any coordinate system that is asymptotically Lorentz. In particular, the gauge need not be Lorenz (or deDonder, i.e. harmonic). For the Schwarzschild metric, written in Schwarzschild coordinates, evaluate this flux integral using the Landau-Lifschitz superpotential [MTW Eq. (20.20), which is equivalent to (20.3) since, in the weak gravity region of the integrand $\sqrt{-g}g^{\mu\nu} = \eta^{\mu\nu} - \bar{h}^{\mu\nu}$].

6. Stress-Energy Tensor for Gravitational Waves [15 Points]

a. Consider a gravitational wave propagating through flat spacetime. Analyze the wave in transverse-traceless (TT) gauge, so the trace-reversed metric perturbation is $\bar{h}^{00} = 0$, $\bar{h}^{0j} = 0$, $\bar{h}_{jk} = h^{\rm TT}_{jk}$. Explain why $h^{\rm TT}_{jj} = 0$ and $h^{\rm TT}_{jk,k} = 0$. (Because the coordinates are Minkowski, it does not matter whether I put spatial indices up or down; repeated spatial indices are to be summed whether up or down.)

b. Show that, to quadratic order in the gravitational-wave field, the Landau-Lifshitz pseudotensor is

$$(-g)t^{\rm LL}_{\mu\nu} = t^{\rm LL}_{\mu\nu} = \frac{1}{32\pi} h^{\rm TT}_{jk,\mu} h^{\rm TT}_{jk,\nu} + (\text{a perfect divergence}) \; . \tag{8}$$

Show, correspondingly, that the stress-energy tensor for the gravitational waves is related to the Landau-Lifshitz pseudotensor by

$$T^{\rm GW}_{\mu\nu} = \frac{1}{32\pi} \langle h^{\rm TT}_{jk,\mu} h^{\rm TT}_{jk,\nu} \rangle = \langle (-g) t^{\rm LL}_{\mu\nu} \rangle = \langle t^{\rm LL}_{\mu\nu} \rangle \; , \tag{9}$$

where $\langle ... \rangle$ denotes an average over a few wavelengths.

7. Law of Forced Precession for a Spinning, Compact Body [20 Points]

Using Newtonian gravity (MTW Exercise 16.4), one can derive a law of precession for the Earth's spin axis, due to coupling of its mass quadrupole moment to the tidal fields of the moon and sun. In this exercise you will show that this same law of precession holds true for any spinning body, no matter how strong or weak its internal gravity may be — even for a black hole. This derivation entails carrying out some details of a computation sketched in Section III of Kip S. Thorne and James B. Hartle, "Laws of Motion and Precession for Black Holes and Other Bodies", *Physical Review D*, **31**, 1815 (1985).

In the body's local asymptotic rest frame, use the Landau-Lifshitz formalism and de-Donder gauge to write the mass-quadrupole part of the body's own gravitational field (its trace-reversed metric perturbation) in the form

$$\bar{h}^{00}_B = \frac{6\mathcal{I}_{jk} n^j n^k}{r^3} \; , \quad \bar{h}^{0j}_B = \bar{h}^{jk}_B = 0 \; . \tag{11}$$

Similarly, write the gravitational field (trace-reversed metric perturbation) of the external universe in the form

$$\bar{h}^{00}_E = -2\mathcal{E}_{jk} x^j x^k \; , \quad \bar{h}^{0j}_E = \bar{h}^{jk}_E = 0 \; . \tag{12}$$

a. Verify that the space-time-space-time components of the Riemann curvature tensor of the external universe, corresponding to Eq. (12), are $R_{j0k0} = \mathcal{E}_{jk}$. In this sense, $\mathcal{E}_{jk}$ is the tidal field that the external universe exerts on the body. In the Newtonian limit, this tidal field is the double gradient of the Newtonian potential, $\mathcal{E}_{jk} = \partial^2\Phi/\partial x^j \partial x^k$.

b. Use the Landau-Lifshitz flux integral for the rate of change of angular momentum,

$$\frac{d\mathcal{S}_i}{dt} = -\int_S \epsilon_{iab} x^a T^{bc} d^2 S_c \tag{13}$$

(where S is a closed surface surrounding the source and lying in its weak-field exterior) to derive the body's law of precession

$$\frac{d\mathcal{S}^j}{dt} = -\epsilon^j{}_{ab}\mathcal{I}^a{}_c\mathcal{E}^{cb} \ . \tag{14}$$

[*NOTE:* You might want to use a computer to evalute the integrand of the flux integral.]

This precession law (8) says that the dot-cross product of the body's mass quadrupole moment with the external tidal field is the tidal torque that acts on the body. In the case where the body is the earth and the external tidal field is the sum of that due to the moon and that due to the sun, the result is a "precession of the equinoxes" (precession of the earth's spin axis) which has a precession period of 26,000 years and was discovered by the ancient Egyptians, perhaps as early as 14,000 BC.

8. **Wave Equations for the Riemann Tensor** [15 Points]

   a. By contracting the Bianchi identity on two slots show that *in vacuum* the Riemann curvature tensor is always divergence free, $R_{\alpha\beta\gamma\delta}{}^{;\alpha} = 0$; and using the symmetries of Riemann show that it is divergence free not only on its first slot, but on all four slots.

   b. By evaluating the divergence of the Bianchi identity and commuting double gradients [making use of the obvious generalization of MTW Eqs. (16.6a,b)], show that *in vaccum* the Riemann curvature tensor always satisfies the following wave equation:

   $$R_{\alpha\beta\gamma\delta;\mu}{}^{\mu} + 2R_{\alpha\beta}{}^{\mu\nu}R_{\delta\mu\gamma\nu} + 2R_{\alpha\mu\delta\nu}R_{\beta}{}^{\mu}{}_{\gamma}{}^{\nu} - 2R_{\beta\mu\delta\nu}R_{\alpha}{}^{\mu}{}_{\gamma}{}^{\nu} = 0 \ . \tag{15}$$

   [Note: I do not guarantee the terms that are products of the Riemann tensor with itself; this is a fairly well known result but I could not find it anywhere in the literature this evening when I searched, so I derived it very quickly for myself and may have made errors.]

   c. Consider gravitational waves propagating through a background spacetime. Split the Riemann tensor into two parts: its average over a few wavelengths, $R^{(\mathrm{B})}_{\alpha\beta\gamma\delta} \equiv \langle R_{\alpha\beta\gamma\delta}\rangle$, and its rapidly varying part $R^{(\mathrm{GW})}_{\alpha\beta\gamma\delta} \equiv R_{\alpha\beta\gamma\delta} - R^{(\mathrm{B})}_{\alpha\beta\gamma\delta}$. Compute the rapidly varying part of the wave equation (15), and estimate the sizes of the various terms. You should have terms whose magnitudes (before the wave equation is actually imposed) are $h/\lambdabar^4$, $h/(\lambdabar^2\mathcal{R}^2)$, and $h^2/\lambdabar^4$, where h is the magnitude of the dimensionless gravitational wave field (the double time integral of Riemann), $\lambdabar$ is the waves' reduced wavelength, and $\mathcal{R}$ is the radius of curvature of the background spacetime.

   d. Show that, aside from tiny fractional errors, your wave equation for $R^{(\mathrm{GW})}_{\alpha\beta\gamma\delta}$ is

   $$R^{(\mathrm{GW})}_{\alpha\beta\gamma\delta;\mu}{}^{\mu} = 0 \ . \tag{16}$$

   This wave equation also follows from the one for $\bar{h}_{\mu\nu}$ in Lorenz gauge, but the above derivation has the lovely feature that one never has to deal with gauges or gauge transformations.

9 Post-Newtonian or Post-Linear Analysis of Collision of two Spinning Black holes

[50 Points — In addition to the maximum of 50 that the rest of the problem set is worth]

I will provide this exercise by Thursday night. It is the one I have been promising: A research problem triggered by numerical relativity results.