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WEEK 10: Geometric Optics, GW Stress-Energy Tensor, and General Relativity as a Nonlinear Field Theory in Flat Spacetime REVISION 2 - ON DECEMBER 6

NOTE: Revision 2 differs from Revision 1 by the addition of the long-promised Problem 9. Revisions 1 include:

A. Ref. 2.d has been expanded to include additional sections in my book.
B. Problem 8: The wave equations you are asked to derive are valid only in vaccum (i.e. when the stress-energy tensor vanishes, i.e. when the Ricci tensor vanishes), so the phrase "in vacuum" has been added, twice.

Recommended Reading:

1. **Stress-energy tensor for gravitational waves**
   a. For a quick derivation, omitting lots of details, along the lines of what I did in class, read Sec. 26.3.4 of Chapter 26 of Blandford and Thorne, version 0626.1.K.pdf; available at http://www.pma.caltech.edu/Courses/ph136/yr2006/ .
   b. For a more detailed treatment, read Sections 35.13–35.15 of MTW.
   c. For the energy and angular momentum carried away from a slow-motion source by gravitational waves, computed as a surface integral of the GW energy flux $T^{0r}_{\rm GW}$, read page 992 of Sec. 36.7 of MTW.
2. **Geometric optics for wave propagation in curved spacetime.**
   a. I suggest beginning with Section 22.5 of MTW. This is a leisurely treatment of geometric optics for quasi-monochromatic electromagnetic waves, giving a lot more detail than I did in my lectures.
   b. Then read (but do not do) MTW Exercise 35.15, which is the analogous geometric-optics treatment of quasi-monochromatic gravitational waves.
   c. Then read Sections 26.3.5 and 26.3.6 of Blandford and Thorne, Chapter 26, version 0626.1.K.pdf . This writes down the geometric optics solution in the time domain in the form I gave in class, and verifies that it satisfies the Einstein field equations.
   d. Finally read Sections 5.A,B,D 5.D of an unpublished 1989 book that I have written on gravitational waves (on our website), which presents and derives the geometric optics equations in the manner that I did in class.
3. **General relativity as a nonlinear field theory in flat spacetime, and its use to derive the evolution laws for the mass, momentum, and angular momentum of a semi-isolated body.**
   a. I suggest beginning by reading MTW chapter 19 ("Mass and Angular Momentum of a Gravitating System") and Secs. 20.1–20.5 (nonlinear field theory and derivation of evolution laws).
   b. Having read MTW, I suggest reading pages 341–344 of Sec. 100 ("The energy-momentum pseudotensor") of Landau and Lifschitz, *The Classical Theory of Fields* (revised second edition, 1962; in a later edition this might be section 101 and different page numbers). This sketches a derivation of the nonliner field theory equations

that I presented in class and that are discussed in MTW — though in a somewhat different notation than I used in class and than MTW.

c. The nonlinear field theory equations and evolution equations are written in a handout that I passed out in class, titled "Landau-Lifshitz Formulation of the Einstein Field Equations". That handout is on our website immediately after this Assignment 10.

Supplementary Reading on GW Stress-Energy Tensor

4. For computation of the energy, linear momentum, and angular momentum carried by gravitational waves, as integrals over a sphere surrounding the source, with the answers expressed as sums over the source's multipole moments, see Thorne, *Reviews of Modern Physics* , **52**, 299 (1980), Sections 4B,C,D (pages 317–319).
5. For a beautiful computation of the energy flux in a gravitational wave based on an analysis of the energy extracted from the wave by a dense collection of mechanical oscillators, see Sec. 9.4 of Schutz, *A First Course in General Relativity*

Supplementary Reading on Geometric Optics:

6. Sections 6.2 and 6.3 of Version 0406.3.K.pdf of Chapter 6 of Blandford and Thorne, available at http://www.pma.caltech.edu/Courses/ph136/yr2004/ . This treats geometric optics for most any type of wave (electromagnetic waves in dielectric media, sound waves in solids such as the interior of the earth, sound waves in fluids, ...).

Supplementary Reading on Applications of the Nonlinear Field Theory Formulation of General Relativity:

7. MTW Sec. 20.6 on "Equations of Motion Derived from Field Equations". This material [due to John Wheeler] describes the conceptual foundations for the derivation of the laws of motion and precession of a self-graviting body (e.g., a black hole) moving through curved spacetime; and it uses those foundations to show that [if one ignores coupling of the body's multipole moments to the external spacetime curvature], the body moves along a geodesic.
8. Kip S. Thorne and James B. Hartle, "Laws of Motion and Precession for Black Holes and Other Bodies", *Physical Review D*, **31**, 1815 (1985). This paper combines the Landau-Lifshitz formalism (general relativity as a field theory in flat spacetime; Refs. 3.b and 3.c above) with the conceptual foundations for laws of motion given in MTW (Ref. 5 above), to derive the actual laws and equations of motion and precession for compact bodies such as neutron stars and black holes.
9. Kip S. Thorne and Yekta Gürsel, "The Free Precession of Slowly Rotating Neutron Stars: Rigid-Body Motion in General Relativity", *Monthly Notices of the Royal Astronomical Society*, **205**, 809 (1983). This paper uses the Landau-Lifshitz formalism to prove that the equations of free precession for a slowly and rigidly rotating body are independent of the strength of the body's internal gravity.

Problems [Each problem is worth 10 points unless otherwise indicated. The maximum number of points you can get from this set is 50 points plus whatever you earn for problem 9.]

1. **Stress-Energy Tensor for a Plane Gravitational Wave in terms of h_+ and $h_\times$**

a. In MTW it is shown that, in any Lorenz gauge ($\bar{h}^{\alpha\beta}_{\ |\beta} = 0$) in which $\bar{h}^{\alpha\beta}$ has been made traceless by a further gauge specialization (e.g., in TT gauge), the wave's stress-energy tensor has the form

$$T^{\rm GW}_{\mu\nu} = \frac{1}{32\pi}\langle \bar{h}_{\alpha\beta|\mu} \bar{h^{\alpha\beta}}_{|\nu}\rangle \ . \tag{6}$$

a. Consider a plane gravitational wave propagating in the z direction as seen in some local Lorentz reference frame. Show that the wave's stress-energy tensor in this case has as its only nonzero components

$$T^{00}_{\rm GW} = T^{0z}_{\rm GW} = T^{z0}_{\rm GW} = T^{zz}_{\rm GW} = \frac{1}{16\pi}\langle \dot{h}_+^2 + \dot{h}_\times^2\rangle \ , \tag{7}$$

where the dots mean $\partial/\partial t$.

b. Show that, in the language of problem 6 of Assignment 8, these waves' energy density and flux have boost weight two (i.e. under a boost along the propagation direction, their magnitudes are changed by the square of the doppler shift factor). Deduce this from the vanishing boost weight of h_+ and $h_\times$ (problem 7.b of Assignment 8, with a typo corrected: the words "spin weight" should be "boost weight"). Also deduce it from the Lorentz transformation law for any stress-energy tensor.

2. Cross Sectional Area of a Bundle of Rays [15 Points]

a. MTW Exercise 22.13.

b. Show that the square root of the cross sectional area $\mathcal{A}$ of a bundle of rays is propagated along the bundle's central ray $\mathcal{C}_0$ according to the equation $\nabla_{\vec{k}}\sqrt{\mathcal{A}} = \frac{1}{2}(\vec{\nabla}\cdot\vec{k})\sqrt{\mathcal{A}}$. This is the same propagation law as for the quantity r that appears in the denominator of the geometric-optics solution $h_{+,\times} = Q_{+,\times}(\tau;\theta,\phi)/r$ that I discussed in my lectures this week and appears in Eq. (26.80) of Blandford and Thorne. Therefore, r is proportional to the square-root of the cross-sectional area of a bundle of rays, which means that the waves' energy flux (which is proportional to $\langle \dot{h}_+^2 + \dot{h}_\times^2\rangle$), dies out as $1/r^2 \propto 1/\mathcal{A}$, as one would expect is required for energy conservation.

3. Gravitons

Read Exercise 35.15 of MTW. Solve Exercise 35.16 of MTW.

4. Gravitational Waves from a Binary Star at the Center of a Spherical Galaxy [15 Points]

A binary star system is at the center of a spherical galaxy, whose background spacetime metric has the standard static, spherical form (same as that for the interior of a spherical star)

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ , \tag{1}$$

with $\Lambda = 0$, $\Phi = \Phi_c$ at the galaxy's center $r = 0$, and $\Lambda \to 0$, $\Phi \to 0$ at radial infinity. Describe the waves' propagation through the galaxy by geometric optics.

a. Because the local wave zone is very small compared to the galaxy and compared to the radius of curvature of spacetime produced by the galaxy's gravity, the metric can be regarded as flat there:

$$ds^2 = -d\hat{t}^2 + d\hat{r}^2 + \hat{r}^2(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{2}$$

(Here I neglect the galaxy's tidal fields and the binary's mass and angular momentum.) The angular coordinates θ and ϕ are the same as those of the galaxy's coordinate system, Eq. (1). What is the relationship between $(\hat{t}, \hat{r})$ and (t, r)?

b. In the local wave zone, introduce gravitational-wave polarization tensors

$$\mathbf{e}_+ = \vec{e}_{\hat{\theta}} \otimes \vec{e}_{\hat{\theta}} - \vec{e}_{\hat{\phi}} \otimes \vec{e}_{\hat{\phi}} \;, \quad \mathbf{e}_\times = \vec{e}_{\hat{\theta}} \otimes \vec{e}_{\hat{\phi}} + \vec{e}_{\hat{\phi}} \otimes \vec{e}_{\hat{\theta}} \;, \tag{3}$$

where $\vec{e}_{\hat{\theta}}$ and $\vec{e}_{\hat{\phi}}$ are unit basis vectors pointing in the θ and ϕ directions. The orientation of the spherical coordinates is chosen so the binary's orbital plane is $\theta = \pi/2$, i.e. it rotates around the $\theta = 0$ axis. The resulting gravitational waves, in the binary's LARF and with the above choice of polarization tensors, have the following waveforms (which you derived last week for an equal-mass binary, but in a slightly different notation):

$$\begin{aligned} h_+ &= 2(1 + \cos^2\theta)\frac{\mu}{\hat{r}}(\pi M f)^{2/3}\cos(2\pi f t) \;, \\ h_\times &= 4\cos\theta\frac{\mu}{\hat{r}}(\pi M f)^{2/3}\sin(2\pi f t) \;. \end{aligned} \tag{4}$$

Here M is the binary's total mass, μ is its reduced mass, and $f = (1/\pi)\sqrt{M/a^3}$ (with a the separation between its stars) is its gravitational-wave frequency. Show that, as seen on the binary's rotation axis, these waves are circularly polarized, while as seen in its orbital plane, they are linearly polarized.

c. These waves propagate out through the galaxy. What are the rays on which they propagate? What are the parallel propagated polarization tensors (you should be able to deduce them from simple symmetry considerations). What are the waveforms measured by an observer far outside the galaxy, who is at rest with respect to the galaxy and the binary?

d. Consider an observer on the binary's rotation axis and far outside the galaxy. This observer is moving toward the galaxy with very high speed v. What are the waveforms measured by this observer?

5. 4-Momentum Flux Integral Applied to Schwarzschild

The flux integral for 4-momentum $P^\mu = \frac{1}{16\pi}\int_S H^{\mu\alpha 0j}{}_{,\alpha} d^2 S_j$ [MTW Eq. (20.9)] is valid in any coordinate system that is asymptotically Lorentz. In particular, the gauge need not be Lorenz (or deDonder, i.e. harmonic). For the Schwarzschild metric, written in Schwarzschild coordinates, evaluate this flux integral using the Landau-Lifschitz superpotential [MTW Eq. (20.20), which is equivalent to (20.3) since, in the weak gravity region of the integrand $\sqrt{-g}g^{\mu\nu} = \eta^{\mu\nu} - \bar{h}^{\mu\nu}$].

6. Stress-Energy Tensor for Gravitational Waves [15 Points]

a. Consider a gravitational wave propagating through flat spacetime. Analyze the wave in transverse-traceless (TT) gauge, so the trace-reversed metric perturbation is $\bar{h}^{00} = 0$, $\bar{h}^{0j} = 0$, $\bar{h}_{jk} = h^{\rm TT}_{jk}$. Explain why $h^{\rm TT}_{jj} = 0$ and $h^{\rm TT}_{jk,k} = 0$. (Because the coordinates are Minkowski, it does not matter whether I put spatial indices up or down; repeated spatial indices are to be summed whether up or down.)

b. Show that, to quadratic order in the gravitational-wave field, the Landau-Lifshitz pseudotensor is

$$(-g)t^{\rm LL}_{\mu\nu} = t^{\rm LL}_{\mu\nu} = \frac{1}{32\pi} h^{\rm TT}_{jk,\mu} h^{\rm TT}_{jk,\nu} + (\text{a perfect divergence}) \ . \tag{8}$$

Show, correspondingly, that the stress-energy tensor for the gravitational waves is related to the Landau-Lifshitz pseudotensor by

$$T^{\rm GW}_{\mu\nu} = \frac{1}{32\pi} \langle h^{\rm TT}_{jk,\mu} h^{\rm TT}_{jk,\nu} \rangle = \langle (-g) t^{\rm LL}_{\mu\nu} \rangle = \langle t^{\rm LL}_{\mu\nu} \rangle \ , \tag{9}$$

where $\langle ... \rangle$ denotes an average over a few wavelengths.

7. Law of Forced Precession for a Spinning, Compact Body [20 Points]

Using Newtonian gravity (MTW Exercise 16.4), one can derive a law of precession for the Earth's spin axis, due to coupling of its mass quadrupole moment to the tidal fields of the moon and sun. In this exercise you will show that this same law of precession holds true for any spinning body, no matter how strong or weak its internal gravity may be — even for a black hole. This derivation entails carrying out some details of a computation sketched in Section III of Kip S. Thorne and James B. Hartle, "Laws of Motion and Precession for Black Holes and Other Bodies", *Physical Review D*, **31**, 1815 (1985).

In the body's local asymptotic rest frame, use the Landau-Lifshitz formalism and de-Donder gauge to write the mass-quadrupole part of the body's own gravitational field (its trace-reversed metric perturbation) in the form

$$\bar{h}^{00}_B = \frac{6\mathcal{I}_{jk} n^j n^k}{r^3} \ , \quad \bar{h}^{0j}_B = \bar{h}^{jk}_B = 0 \ . \tag{11}$$

Similarly, write the gravitational field (trace-reversed metric perturbation) of the external universe in the form

$$\bar{h}^{00}_E = -2\mathcal{E}_{jk} x^j x^k \ , \quad \bar{h}^{0j}_E = \bar{h}^{jk}_E = 0 \ . \tag{12}$$

a. Verify that the space-time-space-time components of the Riemann curvature tensor of the external universe, corresponding to Eq. (12), are $R_{j0k0} = \mathcal{E}_{jk}$. In this sense, $\mathcal{E}_{jk}$ is the tidal field that the external universe exerts on the body. In the Newtonian limit, this tidal field is the double gradient of the Newtonian potential, $\mathcal{E}_{jk} = \partial^2\Phi/\partial x^j \partial x^k$.

b. Use the Landau-Lifshitz flux integral for the rate of change of angular momentum,

$$\frac{d\mathcal{S}_i}{dt} = -\int_S \epsilon_{iab} x^a T^{bc} d^2 S_c \tag{13}$$

(where S is a closed surface surrounding the source and lying in its weak-field exterior) to derive the body's law of precession

$$\frac{d\mathcal{S}^j}{dt} = -\epsilon^j{}_{ab} \mathcal{I}^a{}_c \mathcal{E}^{cb} \ . \tag{14}$$

[*NOTE:* You might want to use a computer to evalute the integrand of the flux integral.]

This precession law (8) says that the dot-cross product of the body's mass quadrupole moment with the external tidal field is the tidal torque that acts on the body. In the case where the body is the earth and the external tidal field is the sum of that due to the moon and that due to the sun, the result is a "precession of the equinoxes" (precession of the earth's spin axis) which has a precession period of 26,000 years and was discovered by the ancient Egyptians, perhaps as early as 14,000 BC.

8. Wave Equations for the Riemann Tensor [15 Points]

a. By contracting the Bianchi identity on two slots show that *in vacuum* the Riemann curvature tensor is always divergence free, $R_{\alpha\beta\gamma\delta}{}^{;\alpha} = 0$; and using the symmetries of Riemann show that it is divergence free not only on its first slot, but on all four slots.

b. By evaluating the divergence of the Bianchi identity and commuting double gradients [making use of the obvious generalization of MTW Eqs. (16.6a,b)], show that *in vaccum* the Riemann curvature tensor always satisfies the following wave equation:

$$R_{\alpha\beta\gamma\delta;\mu}{}^{\mu} + 2R_{\alpha\beta}{}^{\mu\nu}R_{\delta\mu\gamma\nu} + 2R_{\alpha\mu\delta\nu}R_{\beta}{}^{\mu}{}_{\gamma}{}^{\nu} - 2R_{\beta\mu\delta\nu}R_{\alpha}{}^{\mu}{}_{\gamma}{}^{\nu} = 0\ . \tag{15}$$

[Note: I do not guarantee the terms that are products of the Riemann tensor with itself; this is a fairly well known result but I could not find it anywhere in the literature this evening when I searched, so I derived it very quickly for myself and may have made errors.]

c. Consider gravitational waves propagating through a background spacetime. Split the Riemann tensor into two parts: its average over a few wavelengths, $R^{(\mathrm{B})}_{\alpha\beta\gamma\delta} \equiv \langle R_{\alpha\beta\gamma\delta}\rangle$, and its rapidly varying part $R^{(\mathrm{GW})}_{\alpha\beta\gamma\delta} \equiv R_{\alpha\beta\gamma\delta} - R^{(\mathrm{B})}_{\alpha\beta\gamma\delta}$. Compute the rapidly varying part of the wave equation (15), and estimate the sizes of the various terms. You should have terms whose magnitudes (before the wave equation is actually imposed) are $h/\lambda\!\!\!^{-4}$, $h/(\lambda\!\!\!^{-2}\mathcal{R}^2)$, and $h^2/\lambda\!\!\!^{-4}$, where h is the magnitude of the dimensionless gravitational wave field (the double time integral of Riemann), $\lambda\!\!\!^{-}$ is the waves' reduced wavelength, and $\mathcal{R}$ is the radius of curvature of the background spacetime.

d. Show that, aside from tiny fractional errors, your wave equation for $R^{(\mathrm{GW})}_{\alpha\beta\gamma\delta}$ is

$$R^{(\mathrm{GW})}_{\alpha\beta\gamma\delta;\mu}{}^{\mu} = 0\ . \tag{16}$$

This wave equation also follows from the one for $\bar{h}_{\mu\nu}$ in Lorenz gauge, but the above derivation has the lovely feature that one never has to deal with gauges or gauge transformations.

9 Post-Newtonian or Post-Linear Analysis of Collision of two Spinning Black holes [50 Points — In addition to the maximum of 50 that the rest of the problem set is worth]

Read the paper "Maximum Gravitational Recoil" by Manuela Campanelli, Carlos O. Lousto, Yosef Zlochower and David Merritt, *Physical Review Letters* **98**, 231102 (2007);

also available on line at http://xxx.lanl.gov/pdf/gr-qc/0702133 . This paper describes a remarkable discovery made via numerical relativity: an unexpected behavior of two merging black holes that spiral inward toward merger with identical masses and with equal and opposite spins that lie in the plane of their (initially) circular orbit. I propose the following research problem: *Develop a detailed understanding of this behavior using post-Newtonian techniques.*

a. **The pre-merger orbital motion** is displayed in Figures 2 and 3 of Campanelli et. al.: As the circular orbit gradually shrinks, the two holes bob up and down sinusoidally and synchronously — i.e. they move up together and then down together and then up together. You have already explored the inspiral, for nonspinning holes, using Burke's gravitational radiation reaction potential $\Phi^{\rm react}$ in Problem 5 of Week 9. The spins will produce only a small modification of this inspiral—a modification that is not very interesting. Much more interesting is the holes' bobbing. *Use Post-Newtonian Theory to compute this bobbing motion, and compare quantitatively with the numerical results in Figure 3.* I suggest you proceed as follows:

   i. Frans Pretorius, in a lecture in Sydney Australia last August, suggested that this bobbing motion may be due to frame dragging: Each hole drags inertial frames into motion around itself. The other hole, wanting to remain inertial, gets dragged thereby. Show, via diagrams, how this could give rise to the bobbing motion, and compute the magnitude of the bobbing to within factors of order unity.

   ii. It turns out that one can express the influence of frame dragging (i.e. of the g_{0j} components of the metric) on geodesic (inertial) motion via a gravitational analog of the Lorentz force law: g_{0j} acts as a vector potential, whose curl is called the "gravitomagnetic field" $\mathbf{H}$; and the influence of g_{0j} in the geodesic equation can be written in the Lorentz-force form $d\mathbf{v}/dt = \mathbf{v} \times \mathbf{H}$. For details see, e.g., Sec. III.B of Braginsky, Caves and Thorne, *Physical Review D* **15**, 2047 (2007) [Eqs. (3.9) and (3.10)]. By integrating that Lorentz force law over time, for our two black holes, show that the influence of the spin of black hole A on the velocity of hole B is given by

$$\delta v_j^{\rm B} = -\frac{1}{2} g_{0j}^{\rm A} (\text{at location of B}) \ . \tag{17}$$

   (The factor 1/2 might be wrong; I derived it without care. When you have checked and possibly corrected it, try to understand in physical terms why it has the value you compute.) By using the specific form of the gravitomagnetic potential $g_{0j}^{\rm A}$ for a spinning body and assuming a circular orbit, deduce the specific time dependence of the z (up and down) motion of particle B; and similarly for particle A. Show that they bob up and down synchronously, as was seen in the numerical relativity simulations, but *not sinusoidally* as was seen in the simulations. There is a discrepancy between your post-Newtonian analysis and numerical relativity!

   ii. The discrepancy, I think, is due to a second effect, besides frame dragging, that Pretorius did not notice: Spin-curvature coupling, which causes the holes to undergo non-geodesic motion. Read or browse the following paper in order to learn about this spin-curvature coupling and its influence on the equations of motion for the two holes: Thorne and Hartle, "Laws of Motion and Precession for Black Holes and

Other bodies", *Physical Review D* **31**, 1815 (1985). The spin-curvature coupling force appears on the right side of Eq. (1.11a). The binary's post-Newtonian equations of motion, including the influence of the spin-curvature coupling, are derived in Section IV. The terms in the equations of motion that are relevant to us are the ones labeled "(SO)" in Eqs. (4.10), (4.11). See especially Eq. (4.11c) and the two lines of text following it. Compute the influence of these (SO) terms as perturbations of an otherwise circular, nonshrinking orbit. Show that they induce a bobbing motion that *is* sinusoidal. The spin-curvature-coupling force has cancelled out the non-sinusoidal part of the frame dragging.

iii. Next let the orbit shrink in accord with the radiation reaction force and compute the bobbing motion. Compare quantitatively with the numerical relativity results in Figures 2 and 3 of the Campanelli et al paper. You have now gotten as far as I in this analysis. From here onward you will be moving beyond what I have done. Your coordinates are probably different from those used in the numerical simulation. Examine the differences and estimate how strongly they might influence the bobbing details.

iv. The bobbing motion appears to violate momentum conservation: The holes travel up and down together. Explore momentum conservation quantitatively for this binary system, in the simple case of circular orbital motion and sinusoidal bobbing. Do this using the Landau-Lifshitz formulation of general relativity as a nonlinear field theory in flat spacetime — at post-Newtonian order. More specifically, write the total momentum of the binary as

$$P^j = \int (-g)(T^{j0} + t^{j0}_{\rm LL}) d^3x \qquad (18)$$

[MTW Eqs. (10.15) and (20.23a)]. Idealize the black holes (in your Post-Newtonian analysis) as spinning spheres of matter with radii of order the holes' Schwarzschild radius, and thereby incorporate them into T^{j0}. Show that the mechanism for enforcing momentum conservation is that, when the holes bob upward, there is gravitational stress-energy (embodied in $t^{j0}_{\rm LL}$) that bobs downward. Identify where, in the binary system, this counter-bobbing field stress-energy is located.

v. This entire post-Newtonian analysis deals with motion relative to a specific coordinate sytem, so the details are coordinate dependent. Analyze observations made, via light rays, by observers outside the binary. Do they see the holes bob up and down relative to inertial objects on the other side of the binary and far outside it? Can you figure out other coordinate-independent ways to analyze the bobbing? other ways that observers might measure it?

b. **The Kick.** In the numerical simulations by Campanelli et. al., after the holes merge,the combined hole flies upward or downward with a speed (in the case of rapid spins, $a/M \simeq 1$) that can be as large as 4000 km/sec, i.e. ~ 0.01 in geometrized units. This kick speed depends sinusoidally on the binary's orbital phase at merger, though the simulations are not accurate enough to reveal precisely which merger phase gives the big kicks and which gives zero. It is tempting to speculate that this kick is just a continuation of the bobbing: If the holes are rising at merger, they keep flying upward; if they are moving downard at merger, they keep moving downard. And it is tempting to speculate that the burst of gravitational waves that flies off in the opposite direction, to enforce momentum conservation, arises from the counter-bobbing field stress-energy continuing its merger-phase motion and being converted, in the process, into radiation. Use post-Newtonian theory to explore the extent to which this is or is not true, and if it is not the entire story (as I suspect), then what else is going on? I suggest proceeding as follows:

i. Test this speculation via an order-of-magnitude analysis.

ii. The linear momentum carried by the waves arises from a beating of different multipoles against each other; see Eq. (4.20') of my article "Muiltipole Expansion of Gravitational Radiation," *Reviews of Modern Physics*, **52**, 299 (1980). Looking at this formula and thinking about the physics, I expect that the dominant effect will come from the current quadrupole moment beating against the mass quadrupole [third term of Eq. (4.20')]. Do you agree? Why?

iii. Construct a model in which the holes (idealized as spinning point masses) come crashing together in a near head-on collision (their orbital inspiral having become very steep near the end) at some specific phase of the orbit. Compute or estimate the time-changing mass quadrupole and current quadrupole for this collision, and thence the kick force dp_j/dt. Cut the calculation off when the holes are separated by rougly a Schwarzschild radius. Compare the net kick with the simulations. Does it give the same sinusoidal dependence on the merger phase? Is its magnitude the same?

iv. Try to find a way to test the hypothesis that this burst of radiation is produced by a conversion of the counter-bobbing field stress-energy into gravitational waves.

v. There will also be radiation produced by pulsations of the newly merged hole. The following paper estimates this, and also carries out a different variant of the analysis I suggested in (iii): J.D. Schnittman, A. Buonanno et. al., "Anatomy of the Binary Black Hole Recoil: A Multipolar Analysis", http://xxx.lanl.gov/pdf/0707.0301 . Study the portion of this paper that deals with the Campanelli et. al. configuration (identical holes with anti-aligned spins in the orbital plane). Combine whatever you learn there with what you have learned from your own analysis, above, and thereby try to improve your understanding of the kick mechanism.

vi. Do a literature search for other papers, in the past year, that might shed light on what is going on in the Campanelli et. al. simulation. Combine anything you learn there with what you have learned above.

vii. The simulations of this configuration, by Campanelli et. al. and by other finite-difference-based groups, have rather low accuracy and do not show much detail of what is going on in the merger. Fortunately, the ultra-high-precision Caltech/Cornell "spectral" code is now able to simulate mergers: Mark Scheel and grad student Tony Chu, here at Caltech, carried out the first successful simulation of a merger and black-hole ringdown over the past several days — though only for nonspinning, identical holes. It seems likely to me that our group will be able to do a high-precision simulation of the above, spinning binary within the next several months; perhaps sooner. If you have gotten this far in your post-Newtonian analysis, or even only half this far, you will be able to give our numerical relativity group insights into what questions they should ask in their simulations. By going back and forth between your post-Newtonian models and their simulations, you and they may be able to get much deeper insights into what is going on.