## WEEK 22: APPARENT HORIZONS AND SINGULARITIES

**Recommended Reading:**

1. R. M. Wald, *General Relativity* (University of Chicago Press, 1984), Section 12.2 on "General properties of black holes". This discusses trapped surfaces and apparent horizons, and shows in a strongly asymptotically predictable spacetime that satisfies the null energy condition, they are always inside black holes.
2. R. Geroch Appendix D of "Singularities", in *Relativity*, edited by M. Carmelli, S.I. Fickler and L. Witten (Plenum Press, 1970), pages 282–291. This is the singularity theorem whose proof I sketched in class on Monday.
3. R. M. Wald, *General Relativity* (University of Chicago Press, 1984), Chapter 9, "Singularities". This chapter discusses the reasons for defining singularities in terms of geodesic incompleteness of spacetime. It then uses global methods (differential topology), together with various properties of geodesics to prove several singularity theorems.
4. On the BKL (Belinsky, Khalatnikov, Lifshitz) functionally generic singularity: read MTW Sections 30.2, 30.6 and 30.7 (including the long Box 30.1.) Section 30.2 presents the Kasner Solution to Einstein's vacuum field equations for an anisotropic, homogeneous singularity. The BKL spacetime is approximately Kasner during segments of its evolution. Section 30.6 briefly describes the functionally generic, Inhomogeneous and anisotropic BKL singularity structure, including its relationship to Kasner. Section 30.7 focuses in on a homogeneous, anisotropic singularity structure with nonzero spatial curvature (the "Bianchi type IX, Mixmaster Universe") which displays the same sequence of Kasner epochs and eras as BKL—and which, because of its homogeneity, can be analyzed with far greater precision and confidence than BKL. (The Mixmaster solution of the Einstein equations was discovered and explored independently and simultaneously by Misner and by Belinsky, Khalatnikov and Lifshitz around 1970. Misner's presentation of the Mixmaster solution uses elegant Hamiltonian techniques. In the exercises below, I shall follow the more straightforward but less powerful approach of BKL.

   There is no lucid presentation of the BKL inhomogeneous solution that is self contained and reasonably brief. In the suggested supplementary reading below, I give two rather long papers which summarize it.

   The remainder of Chapter 30 (which I have not suggested that you read) focuses on ideas, due to Misner, about how the universe might have been created in a highly inhomogeneous and anisotropic state and then might have been smoothed out by mixmaster oscillations and neutrino viscosity. These ideas, which were popular and pursued with vigor in the early and mid 1970s, ultimately failed. The mixmaster mixing turned out to be insufficient to produce a universe resembling our own. Inflationary cosmology gives a far more satisfactory explanation for the large-scale homogeneity and small-scale inhomogeneity that we observe in the universe today.
5. Partrick Brady, "Some notes on the internal structure of charged black holes", unpublished 1996 notes. This is a pedagogical introduction to the mass-inflation singularity, in

the context of perturbations of the Reissner-Nordstrom spacetime. The situation of real physical interest in perturbations of Kerr spacetime, but because Reissner-Nordstrom is far simpler to deal with (due to its spherical symmetry), most studies of the mass-inflation singularity have been carried out using it rather than Kerr.

6. On the genericity of the mass-inflation singularity: read Amos Ori and Eanna E. Flanagan, "How Generic are Null Spacetime Singularities?" *Phys. Rev. D*, **53**, R1754 (1996); gr-qc/9508066. It also contains references to the literature on the mass-inflation singularity.

**Suggested Supplementary Reading**

7. Thomas A. Roman, "Quantum stress-energy tensors and the weak energy condition", *Phys. Rev. D*, **33**, 3526–3533 (1986); and Thomas A. Roman, "On the averaged weak energy conditon and Penrose's singularity theorem", *Phys. Rev. D*, **37**, 546–548 (1988). These papers show that in Penrose's theorem about singularities inside black holes [Theorem 9.5.3 of Wald], one can replace the weak energy condition by the averaged null energy condition, ANEC (though Roman calls it the averaged weak energy condition). This is important, since quantum fields easily violate the weak energy condition but have great difficulty violating ANEC.
8. These two references, taken together, are a fairly self-contained presentation of the BKL singularity structure:
   a. Vladimir A. Belinsky, Isaac M. Khalatnikov, and Evgeny M. Lifshitz, "Oscillatory Approach to a Singular Point in the Relativistic Cosmology", *Advances in Physics*, **19**, 525–573 (1970).
   b. Vladimir A. Belinsky, Isaac M. Khalatnikov, and Evgeny M. Lifshitz, "A General Solution of the Einstein equations with a Time Singularity", *Advances in Physics*, **31**, 639–667 (1982).
9. David Garfinkle, "Numerical simulations of generic singularities," *Physical Review Letters*, **93**, 161101. This paper describes numerical relativity simulations of the evolution of the spacetime geometry near a generic, spacelike singularity, which verify that the spacetime structure is that of BKL. In the US and Europe there was much skepticism among relativity theorists, in the 1970s, 1980s and 1990s, about the correctness of the BKL analysis. This was because that analysis (references 8 below) was so very complex and had a level of rigor somewhat lower than Western mathematical relativists desired. Garfinkle's simulations have laid that skepticism to rest, I think.

**Problems** [Each problem is worth 10 points unless otherwise stated. The maximum possible number of points on this set is 50.]

1. **Apparent Horizon for Thin Spherical Shell Imploding onto Black Hole**
   As an example of a situation that I described in my Monday lecture: Consider a Schwarzschild black hole with mass $M = M_o$, onto which an arbitrarily thin, spherical shell of photons with mass $M_o$ implodes. By Birchoff's theorem, inside the shell the spacetime will be Schwarzschild with mass $M = M_o$, and outside the shell it will be Schwarzschild with mass $M + 2M_o$. Inside and outside the shell we can introduce

Eddington-Finklestein coordinates $(\tilde{t}, r, \theta, \phi)$:

$$ds^2 = -(1-2M/r)d\tilde{t}^2 + (4M/r)d\tilde{t}dr + (1+2M/r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\ . \quad (1)$$

a. In both coordinate systems adjust the origin of time such that the shell hits the singularity $r = 0$ at $\tilde{t} = \tilde{t}_o$. Then show that the world line of the shell, expressed as $r = r(\tilde{t})$ is identically the same in the interior and exterior coordinates. This means that the coordinates match up perfectly at the shell's world line.

b. Compute the location $r = r_H(\tilde{t})$ of the absolute event horizon in this spacetime and draw it on an Eddington-Finklestein spacetime diagram that includes both the interior and the exterior regions of the spacetime. (You might want to use Mathematica or Maple to do an accurate drawing.)

c. On each 3-surface of constant $\tilde{t}$, identify all apparent horizons. For what values of $\tilde{t}$ are the no apparent horizons, one apparent horizon, and two apparent horizons? Draw these apparent horizons on your spacetime diagram.

**2. Hypersurface Orthogonal Timelike Geodesics** [15 points]

In Sec. 9.2, Wald derives Raychaudhuri's equation [Eq. (9.2.11)] for the rate of change of expansion of a congruence of timelike geodesics. In this exercise we shall specialize this theorem.

Consider a congruence of timelike geodesics that start out orthogonal to some spacelike hypersurface $\mathcal{S}$ and that are all contained in a 3-volume $V$ as they pass through $\mathcal{S}$.

a. Show that the fact that the geodesics are hypersurface orthogonal on $\mathcal{S}$ is equivalent to the statement that their 4-velocity has the form $\vec{u} = -\vec{\nabla}t/|\vec{\nabla}t|$ for some scalar field $t$ that is constant on $\mathcal{S}$ and increases as one moves forward in time off $\mathcal{S}$. (One can think of $t$ as a time coordinate.)

b. Show that this concept of hypersurface orthogonality is also equivalent to the vanishing of the congruence's rotation tensor $\omega_{\alpha\beta} = 0$ (cf. the discussion of Frobenius's theorem at the end of Sec. B.2 of Appendix B of Wald.)

c. Show that, as the geodesics move forward and backward in time away from the hypersurface $\mathcal{S}$, $\omega_{\alpha\beta}$ remains zero, and the geodesics therefore continue to be hypersurface orthogonal.

d. Let $V(\tau)$ be the 3-volume occupied by this bundle of geodesics at proper time $\tau$, as measured by observers who move along the geodesics and who therefore regard the orthogonal hypersurfaces as slices of simultaneity. Show that Raychaudhuri's equation (9.2.11) is equivalent to the following "focusing equation" for the volume $V$:

$$\frac{d^2V^{1/3}}{d\tau^2} = -\frac{1}{3}(\sigma_{\alpha\beta}\sigma^{\alpha\beta} + R_{\alpha\beta}u^\alpha u^\beta)V^{1/3}\ . \quad (1)$$

e. Show that, if the weak energy condition is satisfied, $R_{\alpha\beta}\xi^\alpha\xi^\beta \geq 0$ for all causal $\vec{\xi}$, then $d^2V^{1/3}/d\tau^2 \leq 0$.

f. Suppose that $\theta \equiv \vec{\nabla}\cdot u = V^{-1}dV/d\tau$ has a negative value $\theta_o < 0$ at $\tau = 0$. Show from Eq. (1) that the volume $V$ decreases to zero in a proper time no longer than $\Delta\tau = 3/|\theta_o|$. This is a more physical point of view on Proposition 9.3.4 of Wald.

**3. Conjugate Point to a Spacelike Hypersurface**
In proving a singularity theorem on Monday, I needed the following theorem. Prove it. Consider the congruence of timelike geodesics that all leave a spacelike hypersurface $\Sigma$ orthogonally, traveling forward in time. (Thus, these geodesics will be the world lines of observers who all regard $\Sigma$ as a slice of simultaneity.) Suppose that two of these geodesics $\gamma_1$ and $\gamma_2$, which start out with arbitrarily small (proper length) separation $l$ on $\Sigma$, cross at some event $\mathcal{P}$. Show that the proper times from $\Sigma$ to $\mathcal{P}$ along $\gamma_1$ and $\gamma_2$ are the same through second order in their initial separation $l$, i.e. $\tau_1 - \tau_2 = \mathcal{O}(l^3)$.
*Hint, if you need it:*

This result is a close analog of the first six lines of the proof of Theorem 9.3.3 on page 229 of Wald.

**4. Synchronous Coordinates** [5 points]
Choose an initial spacelike hypersurface $\mathcal{S}_0$ in spacetime and on it introduce an arbitrary set of spatial coordinates $x^j$. From each event on $\mathcal{S}_0$ and in a direction orthogonal to $\mathcal{S}_0$, launch a future-directed timelike geodesic. Carry the spatial coordinates along these geodesics, and use as the time coordinate $t$ the proper times along these geodesics, measured from $t = 0$ at $\mathcal{S}_0$. Show that in this coordinate system the spacetime metric takes on the synchronous form

$$ds^2 = -dt^2 + \gamma_{ij} dx^i dx^j \ , \tag{2}$$

where $\gamma_{ij}$ are functions of $(t, x^j)$.

**5. BKL Evolutionary Equations** [20 points]
The BKL metric (Refs. [8]) has the synchronous form (2) with

$$\gamma_{ij} = a^2 l_i l_j + b^2 m_i m_j + c^2 n_i n_j \ . \tag{3}$$

Here (i) during each Kasner epoch, $l_i$, $m_i$ and $n_i$ are functions of $x^k$ only (and not of time $t$), but during the transition from one epoch to another, they change somewhat (a phenomenon that BKL call "rotation of the Kanser axes" [8]); (ii) during each Kasner epoch, $a$, $b$, and $c$ have the Kasner form

$$a = t^{p_l} \ , \quad b = t^{p_m} \ , \quad c = t^{p_n} \ , \tag{4}$$

with $p_l$, $p_m$, and $p_n$ functions of $x^k$ but not of $t$; and between epochs $a$, $b$, and $c$ evolve differently from this.

The evolutions of the Kasner exponents $p_l$, $p_m$ and $p_n$ from one epoch to the next (and one era to the next) are governed by the diagonal components of the Einstein field equations, Eqs. (5) below. We shall study those evolutions in this exercise and the next. The rotations of the Kasner axes from one epoch to the next (and one era to the next) are governed by the off-diagonal components of the field equations; see Sec. 3 of Ref. [8b] and references therein. We shall not study these rotations.

The Einstein field equations for $a$, $b$, and $c$, both during Kasner epochs and during the transitions between epochs, take the following forms [cf. Eqs. (3.14) of Ref. [5] as extended by comparing with Eqs. (3.1) and (3.2) of Ref. [8a]]:

$$-R^l_l = \frac{(a_{,t}bc)_{,t}}{abc} + \frac{1}{2a^2b^2c^2}[\lambda^2 a^4 - (\mu b^2 - \nu c^2)^2] = 0 \ , \tag{5a}$$

$$-R^m_m = \frac{(ab_{,t}c)_{,t}}{abc} + \frac{1}{2a^2b^2c^2}[\mu^2 b^4 - (\lambda a^2 - \nu c^2)^2] = 0 \ , \tag{5b}$$

$$-R^n_n = \frac{(abc_{,t})_{,t}}{abc} + \frac{1}{2a^2b^2c^2}[\nu^2 c^4 - (\lambda a^2 - \mu b^2)^2] = 0 \ , \tag{5c}$$

$$-R^0_0 = \frac{a_{,tt}}{a} + \frac{b_{,tt}}{b} + \frac{c_{,tt}}{c} = 0 \ . \tag{5d}$$

Here

$$\lambda = \frac{\vec{l}\cdot(\vec{\nabla}\times\vec{l})}{\vec{l}\cdot(\vec{m}\times\vec{n})} \ , \quad \mu = \frac{\vec{m}\cdot(\vec{\nabla}\times\vec{m})}{\vec{m}\cdot(\vec{n}\times\vec{l})} \ , \quad \nu = \frac{\vec{n}\cdot(\vec{\nabla}\times\vec{n})}{\vec{n}\cdot(\vec{l}\times\vec{m})} \ , \tag{6}$$

where the notation is that of flat-space vector analysis; i.e., in Eqs. (6) the dot, cross and curl are evaluated as though the coordinates $x^k$ were Cartesian. These evolutionary equations are the same as for the homogeneous Mixmaster Cosmology described in Ref. [2], except for the specific values of $\lambda$, $\mu$, $\nu$: For Mixmaster, $\lambda = \mu = \nu = 1$; for BKL, they are spatially variable.

a. Show that, when the second terms on the right-hand side of (5a,b,c) [the terms with square brackets] are negligible, the solution of these equations has the Kasner form (4) with the $p$'s satisfying the Kasner relations

$$p_l + p_m + p_n = 1 \ , \quad p_l^2 + p_m^2 + p_n^2 = 1 \ . \tag{7}$$

b. Fix $x^j$ and examine the evolution of $(a, b, c)$ with *decreasing* time t during one Kasner epoch, from the end of one transition to the beginning of the next. Suppose that during this Kasner epoch $p_l < 0 < p_m < p_n$. During the previous epoch, the ordering was $p_m < 0 < p_l < p_n$. Correspondingly, at the beginning of this epoch $b$ is large (it was driven to a large value by expansion along the $\vec{m}$ direction of the previous epoch), $a$ is small and $c$ is even smaller. Show that as spacetime moves out of the transition, its evolution becomes more and more nearly Kasner for awhile and then the Kasner approximation begins to break down as the next transition is neared. Compute the duration of the Kasner epoch in terms of the initial values of $(a, b, c, t)$.

c. Analyze the transition at the end of this Kasner epoch. Do so by changing variables from $t$ to $\tau$ given by $dt = abcd\tau$ and from $(a, b, c)$ to $\alpha = \ln a$, $\beta = \ln b$, $\gamma = \ln c$. Neglecting the $\mu$ and $\nu$ terms in the evolutionary equations, show that the transition is to a new Kasner epoch with

$$p'_l = \frac{-p_l}{1+2p_l} \ , \quad p'_m = \frac{2p_l + p_m}{1+2p_l} \ , \quad p'_n = \frac{p_n + 2p_l}{1+2p_l} \ . \tag{8}$$

It is easy to see that this corresponds to the mapping described in the next problem.

d. Consider a succession of Kasner epochs, with transitions of the form described in the next problem. Analyze and discuss the durations of these epochs (measured in terms of proper time $t$).

**6. The BKL Map**

The BKL map, from one Kasner epoch to another and from one Kasner era to another, is as follows: (i) Define $p_1(u)$, $p_2(u)$, and $p_3(u)$ by

$$p_1 = \frac{-u}{1+u+u^2}\ , \quad p_2 = \frac{1+u}{1+u+u^2}\ , \quad p_3 = \frac{u(1+u)}{1+u+u^2}\ , \tag{9}$$

with $u > 1$ some irrational number. (ii) Before the transition, $(p_l, p_m, p_n)$ are equal to some permutation of $(p_1, p_2, p_3)$. (iii) If $u > 2$, then the transition is an ordinary one, from epoch to epoch, and in the transition $u$ changes to $u - 1$, and the 1 and 2 directions exchange places (so, e.g., if $p_m = p_1$ and $p_l = p_2$ before the transition, then $p_l = p_1$ and $p_m = p_2$ afterward). (iv) If $1 < u < 2$, then the transition is the end of one era and the beginning of the next, and in the transition $u$ changes to $1/(u-1)$, and the 2 and 3 directions exchange places.

Write a computer program that implements this map, and use it to explore the chaotic behavior of the map. Number the epochs 1, 2, 3, ... . Plot $p_l$, $p_m$ and $p_n$ as a function of epoch number for enough epochs to include a fairly large number of eras.

**7. The Vaidya Metric**

The Vaidya metric is a generalization of Reissner-Nordstrom to include a radiative influx of energy in the geometric optics limit. It is the foundation for Ori's simple model of the mass-inflation singularity, as expounded in Ref. [5] above. The relevant variant of the Vaidya metric has the following form:

$$ds^2 = -\left(1 - \frac{2m(v)}{r} + \frac{Q^2}{r^2}\right) dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2)\ , \tag{1}$$

where $Q$ is a constant and $m$ is a monotonically increasing function of $v$ — for example, $m$ could be

$$m = (2M/\pi)[\pi + \arctan(v/M)]\ , \tag{2}$$

which increases monotonically from $M$ to $3M$.

a. Show that, if $m(v)$ has the form (2), then at early times this metric describes a Reissner-Nordstrom black hole with mass $M$ and charge $Q$, and at late times it is a Reissner-Nordstrom black hole with mass $3M$ and charge $Q$.

b. Evaluate the stress-energy tensor for this spacetime assuming general $m(v)$. Show that it consists of two parts, the stress-energy of a radial electric field with total charge $Q$, and that of geometric optics radiation which streams inward at the speed of light. What is the total inflowing luminosity as measured by a distant observer at location $(v, r, \theta, \phi)$?

c. Draw an Eddington-Finkelstein type of spacetime diagram for this metric, plotting $\tilde{t} = v + r$ upward and $r$ horizontally so that inward radial light rays are tilted at a 45 degree angle to the vertical axis. Compute the following quantities and plot them, for the case of $m(v)$ given by Eq. (2): (i) The location of the apparent horizon as seen on slices of constant $\tilde{t}$; (ii) The location of the absolute event horizon.