Ph 236c: General Relativity 23 April 2008

WEEK 23: Thin Shells and Braneworlds

Recommended Reading:

1. MTW: The following materials dealing with junction conditions across 3-surfaces in curved 4-dimensional spacetime:
   a. Sec. 21.5 : The intrinsic geometry (metric) and extrinsic curvature of a 3-surface in 4-dimensional spacetime; the Gauss-Codazzi equations expressing the 4-dimensional Riemann tensor in terms of the surface's intrinsic and extrinsic geometries and the normal derivative of the extrinsic geometry; and the use of the Gauss-Codazzi equations to derive equations for the Einstein tensor of 4-dimensional spacetime in terms of the surface's intrinsic and extrinsic geometries and the normal derivative of the extrinsic geometry. Notes: (i) this section assumes that the 3-surface is spaceline; in our of thin shells and braneworlds this weak it is timelike. (ii) This section relies to some extent on Sec. 21.4, which discusses the 3+1 split of the spacetime metric into 3-metric, lapse function and shift function. We shall return to these issues next week, when we discuss the dynamics of geometry and numerical relativity. (iii) As I discussed in my lecture on Monday, the Gauss-Codazzi equations and Einstein tensor derived in this section are valid in a spacetime of any dimension, not just 4 and a surface (brane) with one lower dimension, living in that spacetime. (v) For an alternative treatment of this material see Ref. [4] below.
   b. Sec. 21.13, which summarizes the equations for the split of the 4-D Einstein tensor into contributions from the intrinsic and extrinsic curvatures of a hypersurface, and then uses those equations analyze a thin shell. Again, this analysis is valid in any any dimension — except when one the trace of a metric and thereby gets a dimension-dependent number.
2. T. Shiromizu, K-i Maeda and M. Sasaki, "The Einstein Equations on the 3-brane World", *Phys. Rev. D*, **62**, 127502 (2000); also available at http://arXiv.org/abs/hep-th/0007203 . This paper is the original analysis of our universe as a brane in 5-dimensional spacetime — the analysis that I sketched in my lecture on Wednesday. It is readable and fairly short.
3. R. Maartens, "Brane-World Gravity," *Living Reviews in Relativity* **7**, (2004), 7. URL http://www.livingreviews.org/lrr-2004-7 : Secs. 1–3.1 (pages 1–17). This is a very nice review article on braneworlds. The sections I'm assigning include motivational material on the hierarchy problem, a summary of the two Randall-Sundrum models, and then a derivation of the brane's 4-dimensional Einstein equations from the 5-dimensional Einstein equations of the bulk, along the lines of my Wednesday lecture and Ref. [2].

Supplementary Reading:

4. Robert M. Wald, *General Relativity*, Sec. 10.2 "Initial Value Formulation of General Relativity". This covers much the same ground as as Chap. 21 of MTW; ref. [1].

5. L. Randall and R. Sundrum, "An Alternative to Compactification", *Phys. Rev. Lett.*, **83**, 4690 (1999). This paper introduces the RS-1 model with just one brane (our universe) in a 5-dimensional Anti-deSitter-space bulk — the model that I "derived" in my Wednesday lecture.
6. L. Randall and R. Sundrum, "Large Mass Hierarchy from a Small Extra Dimension", *Phys. Rev. Lett.*, **83**, 3370 (1999). This paper introduces the RS-2 model with two branes separated by a sandwich of 5-dimensional Anti-deSitter spacetime.
7. Lisa Randall, *Warped Passages* (Harper Collins, New York, 2005). A popular book about braneworlds.

Problems [Each problem is worth 10 points unless otherwise stated. The maximum number of points for this problems set is 50.]

1. **Imploding Ball of Dust Treated by Joining a Segment of a Closed Friedmann Universe onto the Schwarzschild Solution**
   Read Sec. 32.4 of MTW. Then work Ex. 32.4 on page 853. Note: Chap. 32 is on our course website — put there in first term.
2. **Equation of Motion for a Surface Layer**
   MTW Exercise 21.25. Note that this discusses only the surface layer's internal motion and not its motion relative to the surrounding space.
3. **Normal Force Balance for a Surface Layer**
   This exercise extends the result of Exercise 1 to include motion of the surface layer in the surrounding spacetime.
   a. Show, from the equations in Sec. 21.13 of MTW, that

   $$\frac{1}{2}(K_+^{ab} + K_-^{ab})S_{ab} = -[T^{nn}] \ , \tag{1}$$

   where K_+^{ab} is the extrinsic curvature on the + side of the surface layer and K_-^{ab} is the extrinsic curvature on the − side. Discuss the physical meaning of this equation.

   b. Let $\vec{u}$ be the 4-velocity of the matter in the surface layer, as defined in Exercise 1, and let $\vec{a}_\pm \equiv \nabla_{\vec{u}}\vec{u}$ be the 4-acceleration of that matter as measured from the two sides of the surface layer. Then the a_j of Exercise 1 is the projection of $\vec{a}_\pm$ into the surface layer (and is continuous so we don't need any ± subscript). Denote by $a_\pm \equiv \vec{a}_\pm \cdot \vec{n}$ the projection of the 4-acceleration along the surface layer's normal direction. Do a 2+1 split in the surface layer, using an orthonormal basis $e_{\hat{0}} = \vec{u}$, $\vec{e}_{\hat{p}}$, with p and q ranging over 1 and 2. From the fact that $\vec{u} \cdot \vec{n} = 0$, show that

   $$K_{\pm\ \hat{0}\hat{0}} \equiv \vec{u} \cdot \mathbf{K}_\pm \cdot \vec{u} = a_\pm \ . \tag{2}$$

   c. Show that

   $$\frac{1}{2}\sigma(a_+ + a_-) = -\frac{1}{2}t_{\hat{p}\hat{q}}(K_-^{\hat{p}\hat{q}} + K_+^{\hat{p}\hat{q}}) - [T^{nn}] \ . \tag{3}$$

   Discuss the physical interpretation of this equation, as an equation of motion for the surface layer in the surrounding spacetime.

4. Vilenkin's Static Domain Wall

Alexander Vilenkin has solved Einstein's equations for a static domain wall with a cosmological-constant stress-energy tensor $S^{ab} = -\sigma q^{ab}$, with q^{ab} the domain wall's spacetime metric. His claimed solution has the following 4-dimensional (bulk) metric:

$$ds^2 = -(1-\kappa|z|)^2 dt^2 + dz^2 + (1-\kappa|z|)^2 e^{2\lambda t}(dx^2 + dy^2) \ . \tag{4}$$

Here κ is a constant, and the domain wall is at $z = 0$.

a. What is the domain wall's mass per unit area and what is its surface tension?

b. Compute the extrinsic curvature of the wall's world tube $z = 0$ as seen on its two sides, and show that $\gamma_{ab} \equiv [K_{ab}]$ satisfies the junction conditions $8\pi S_{ab} = \gamma_{ab} - \gamma^c_c q_{ab}$ if κ has a particular value, proportional to the wall's mass per unit area σ. What is that value?

c. Show that the law of perpendicular force balance, Eq. (3), is also satisfied, as are the laws of 3-momentum conservation in the wall, Eqs. (21.172) and (21.173) of MTW.

c. Having worked this problem, then read or at least browse Vilenkin's paper, to learn the context and significance of this problem: Alexander Vilenkin, "Gravitational Field of Vacuum Domain Walls", *Physics Letters B*, **133B**, 177–179 (1983).

5. Expanding Bubble Wall [15 points]

Suppose that a spherical bubble of new vacuum, with "cosmological constant" $\Lambda = 0$, forms inside the inflationary vacuum, which has $\Lambda > 0$ and $T_{\mu\nu} = -\rho_\Lambda g_{\mu\nu}$ with $\rho_\Lambda = \Lambda/8\pi$. The new "Minkowski" vacuum and the old "deSitter" vacuum are separated by a domain wall with surface stress-energy tensor $S_{ab} = -\sigma q_{ab}$, where q_{ab} is the 3-metric of the domain wall. In the interior of the domain wall, spacetime is flat so the line element has the Minkowski form $ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$; and the wall's world line as seen in its Minkowski interior is $r = R(t)$. Outside the domain wall (the old vacuum) the spacetime metric is that of a deSitter universe with a Schwarzschild solution (produced by the domain wall's mass) embedded in it, i.e. the "Schwarzschild-deSitter metric"; see, e.g., G.W. Gibbons and S.W. Hawking, *Phys. Rev. D*, **15**, 2738 (1977).

a. Show that as the bubble expands, its surface mass density σ (equal to surface tension) remains constant, independent of time.

b. Using the equation of motion (3) and other equations from Sec. 21.13 of MTW, derive expressions for a_+ and a_- in terms of ρ_Λ, σ and R.

c. Show that, as the bubble expands, its acceleration a_- grows, approaching a limiting uniform acceleration at large R. Show that at late times the bubble wall is moving outward at nearly the speed of light. Draw a graph of $R(t)$.

d. Discuss qualitatively the expected motion of the domain wall as seen on its + side.

6. Weyl Tensor [5 points]

In a spacetime with metric $G_{\alpha\beta}$ and dimension N, the Riemann tensor can be written as a sum of terms proportional to the the Ricci scalar, the Ricci tensor, and the Weyl tensor $C_{\mu\alpha\nu\beta}$ — which is trace-free on all pairs of slots:

$$R_{\mu\alpha\nu\beta} = Ag_{\mu[\nu}g_{\beta]\alpha}R + B(g_{\mu[\nu}R_{\beta]\alpha} - g_{\alpha[\nu}R_{\beta]\mu}) + C_{\mu\alpha\nu\beta}\ . \tag{5}$$

Derive the values of the coefficients A and B in terms of the spacetime's dimension N. For $N = 5$ your answer should agree with the formula (7) used by Shiromizu et. al. (Ref [2]) in their braneworlds analysis.

7. Orders of Magnitude for RS 1-Brane Model of our Universe

Near the end of my Wednesday lecture, a student question suggested I had made a mistake in my order of magnitude estimates for the RS 1-brane model. I indeed had made a mistake. In this exercise you will work out the estimates for yourself, beginning with the relation ${}^{(4)}\kappa^2 = \kappa^2/\ell$ between the coupling constant ${}^{(4)}\kappa^2 = 8\pi G_{\text{Newton}}$ appearing in our universe's Einstein equations, the coupling constant κ^2 appearing in the bulk's five-dimensional Einstein equations, and the warp length ℓ of the bulk's anti-deSitter metric.

a. The Planck lenghths ${}^{(4)}L_P$ and L_P of our 4-dimensional brane and of the 5-dimensional bulk are the lengthscales that one can construct by algebraically combining their gravitational coupling constants with Planck's constant $\hbar$; and similarly the Planck masses are those ${}^{(4)}M_P$ and M_P constructable by combining the gravitational coupling constants with $\hbar$. Show by dimensional analysis that, up to a factor of order unity (which depends on precisely how one defines the Planck lengths and masses),

$$L_P^3 = \kappa^2\hbar\ , \quad {}^{(4)}L_P^2 ={}^{(4)}\kappa^2\hbar\ ; \quad M_P = \frac{\hbar}{L_P} \quad {}^{(4)}M_P = \frac{\hbar}{{}^{(4)}L_P}\ . \tag{6}$$

Evaluate numerically the 4-dimensional Planck length in centimeters and Planck mass in TeV.

b. Show that

$${}^{(4)}L_P = L_P\left(\frac{{}^{(4)}Lp}{\ell}\right)^{1/3}\ , \quad {}^{(4)}M_P = M_P\left(\frac{\ell}{{}^{(4)}L_p}\right)^{1/3}\ . \tag{7}$$

c. Inverse-square-law experiments by former Blacker-House Head Waiter Eric Adelberger and colleagues [*Phys. Rev. Lett.* **98**, 021101 (2007)] show that $\ell < 56\mu m$. What limits does this then place on the bulk's Planck length and Planck mass (which are thought of as the "fundamental" lengths and masses associated with gravity). While this limit does not allow the fundmental Planck mass to be as low as the energy scales for other fundamental forces, $\sim$ 1 TeV, it allows M_P to be a lot closer to 1 TeV than is ${}^{(4)}M_P$.

d. Show that, if ρ is the mass density in our brane, then in order of magnitude, the ratio of the quadratic source term in the 4-dimensional Einstein equations to the usual linear source term is

$$\frac{\kappa^4\pi^{ab}}{{}^{(4)}\kappa^4 T^{ab}} \sim G_{\text{Newton}}\ell^2\rho \sim \frac{\rho}{10^{28}\text{g cm}^{-1}\ell^{-2}}\ , \tag{8}$$

where the notation is that of my Wednesday lecture and of Eqs. (17)–(20) of Shiromizu et. al. [2]. Using Adelberger's experimental limit on the Anti-deSitter warp length, show that the quadratic source term is neglible so long as the mass density in our universe is $\rho \ll 10^{32}\mathrm{g/cm}^3$. How does this compare with the density inside a neutron star (the densest object that occurs in our universe today)?

8. Braneworld with AdS Sandwich Living in Flat 5-Dimensional Spacetime Gregory, Rubakov and Sibiryakov, hep-th/0002072 , introduced a braneworld model in which our universe is a 4-dimensional brane surrounded by thin layers of 5-dimensional AdS bulk (warp length ℓ and thickness L), which in turn are sandwiched between branes that live in flat 5-dimensional spacetime. More specifically, let w be a coordinate that measures distance travelled orthogonally off of our brane. (I used the notation n in my lecture, as in MTW.) The 5-dimensional spacetime metric in the sandwich is that of AdS

$$ds^2 = e^{-2|w|/\ell}\eta_{ab}dx^a dx^b + dw^2 \quad \text{at } 0 \le |w| \le L \,, \tag{9a}$$

and it is flat spacetime outside the sandwich

$$ds^2 = \eta_{ab}d\bar{x}^a d\bar{x}^b + dw^2 \quad \text{at } |w| \ge L \,, \quad \text{where } \bar{x}^a \equiv e^{-L/\ell}x^a \,. \tag{9b}$$

a. Compute explicitly the extrinsic curvatures (on its two sides) of our brane at $w = 0$ and also the extrinsic curvatures of the two "confining" branes at $w = -L$ and $w = +L$. Then from the Israel junction conditions compute the stress-energy tensors on each of the three branes. Your answers should be $S_{ab} = -\lambda\eta_{ab}$ on our brane and $S_{ab} = -\lambda_c\eta_{ab}$ on the confining branes, with mass per unit volume (equal to brane tension)

$$\lambda = \frac{6}{\kappa^2\ell} \,, \quad \lambda_c = -\frac{3}{\kappa^2\ell} \,. \tag{10}$$

Thus, the confining branes have negative mass density and negative tension (positive pressure). This suggests that the the confining branes will be unstable to buckling. (Perturb their location $w = \pm L$ slightly, in an inhomogenous way, and those perturbations will grow). This, indeed, is the case; see Pilo, Rattazzi and Zaffaroni, hep-th/0004028 . A similar phenomenon was thought, initially, to occur in the RS 2-brane models, until Goldberger and Wise, hep-ph/9911457, showed that the negative-tension brane could be stabilized by a radion field. In the above model, as I understand it (and I am not an expert), the fact that one side of the confining brane is flat spacetime instead of both sides being AdS prevents the Goldberger-Wise stabilization mechanism from working.

b. Explain why, if the confining branes could be stabilized, and if $L \gg \ell$, this model would give the same predictions for 4-dimensional gravitational physics as the RS 1-brane model.

c. By thinking about the Newtonian-gravity limit as discussed in my Wednesday lecture and by Maartens [3], explain why, if $L \sim \ell$ or smaller, this model will *not* give the same predictions for 4-dimensional gravitational physics as the RS 1-brane model.

d. Explain why, even if $L \sim \ell$ or smaller, the 4-dimensional Einstein equations for our brane will be identical to those of the RS 1-brane model. Explain how, despite this, gravitational physics on our brane will *not* be the same as for RS-1brane. [Hint: What is the role of the bulk's Weyl tensor?]