Ph 236c: General Relativity

WEEK 24: 3+1 Spacetime Decomposition

Due: May 7, 2008

Suggested Reading:

1. Baumgarte and Shapiro, “Numerical Relativity and Compact Binaries,” *Physics Reports*, **376**, 41–131 (2003); gr-qc/0211028.
2. MTW (*Gravitation*): Chapter 21.
3. Wald (*General Relativity*): Chapter 10.

Problems

1. **3+1 Metric.** a) Show that the inverse of the standard 3+1 metric,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt),$$

may be written as

$$ds^{-2} = g^{\mu\nu}\partial_\mu \partial_\nu = -\alpha^{-2}(\partial_t - \beta^i \partial_i)^2 + \gamma^{ij}\partial_i \partial_j,$$

where γ^{ij} is the inverse of γ_{ij}.

b) Consider a spacelike surface Σ that is embedded in a four-dimensional spacetime, i.e., Σ is the level surface of a “time function” t. Let n^μ be the unit vector that is orthogonal to Σ, i.e. n_μ is proportional to $\nabla_\mu t$. Define the three dimensional metric $\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$, where $g_{\mu\nu}$ is the spacetime metric. Show that in the coordinates of a) that the components of $\gamma_{\mu\nu}$ are $\gamma_{tt} = \beta^i \beta_i$, $\gamma_{ti} = \beta_i$, and $\gamma_{ij} = \gamma_{ij}$.

c) Define a covariant derivative D_μ that acts only on vectors that are tangent to the surface Σ: let

$$D_\mu X_\nu = \gamma^\alpha{}_\mu \gamma^\beta{}_\nu \nabla_\alpha X_\beta,$$

for X_μ that satisfy $X_\mu n^\mu = 0$, where ∇_α is the covariant derivative compatible with the spacetime metric $g_{\alpha\beta}$. Show that D_μ can be thought of as the three-dimensional covariant derivative intrinsic to the surface Σ by showing that $D_\alpha \gamma_{\mu\nu} = 0$. It follows that D_α is the unique covariant derivative that is compatible with the metric $\gamma_{\mu\nu}$.

d) Define the extrinsic curvature of Σ as $K_{\mu\nu} = -\gamma^\alpha{}_\mu \nabla_\alpha n_\nu$. Show that $K_{\alpha\beta} = K_{\beta\alpha}$ and that $K_{\alpha\beta} n^\beta = 0$. Also show that $K_{\mu\nu}$ can be written as $K_{\mu\nu} = -\frac{1}{2}\mathcal{L}_n \gamma_{\mu\nu}$, where the Lie derivative $\mathcal{L}_n$ is defined as $\mathcal{L}_n Q_{\mu\nu} = n^\alpha \nabla_\alpha Q_{\mu\nu} + Q_{\alpha\nu}\nabla_\mu n^\alpha + Q_{\mu\alpha}\nabla_\nu n^\alpha$.

2. Define the curvature $R^\alpha{}_{\beta\mu\nu}$ associated with the surface Σ by using the Ricci identity:

$$D_\mu D_\nu X_\beta - D_\nu D_\mu X_\beta = -R^\alpha{}_{\beta\mu\nu} X_\alpha,$$

for $X_\mu n^\mu = 0$, and $R^\alpha{}_{\beta\mu\nu} n_\alpha = 0$. Express the covariant derivatives on the left side of this definition in terms of ∇_μ to show that

$$R_{\alpha\beta\gamma\delta} = \gamma^\mu{}_\alpha \gamma^\nu{}_\beta \gamma^\sigma{}_\gamma \gamma^\tau{}_\delta {}^{(4)}R_{\mu\nu\sigma\tau} - K_{\alpha\gamma}K_{\beta\delta} + K_{\alpha\delta}K_{\beta\gamma},$$

where ${}^{(4)}R_{\mu\nu\sigma\tau}$ is the four-dimensional Riemann curvature associated with ∇_α.

3. Use the four dimensional Ricci identity

$$\nabla_\mu \nabla_\nu n_\beta - \nabla_\nu \nabla_\mu n_\beta = -{}^{(4)}R^\alpha{}_{\beta\mu\nu} n_\alpha,$$

to show that

$$\gamma^\mu{}_\alpha \gamma^\nu{}_\beta \gamma^\sigma{}_\gamma n^\tau \; {}^{(4)}R_{\mu\nu\sigma\tau} = D_\beta K_{\alpha\gamma} - D_\alpha K_{\beta\gamma},$$

and

$$\gamma^\alpha{}_\mu n^\beta \gamma^\sigma{}_\nu n^\tau \; {}^{(4)}R_{\alpha\beta\sigma\tau} = \mathcal{L}_n K_{\mu\nu} + \alpha^{-1} D_\mu D_\nu \alpha + K_\mu{}^\alpha K_{\alpha\nu}.$$

4. Use the results of Problems #2 and #3 to show that:

$$n^\mu n^\nu \; {}^{(4)}G_{\mu\nu} = \frac{1}{2}\big(R + K^2 - K_{\mu\nu}K^{\mu\nu}\big),$$

$$n^\mu \gamma^\nu{}_\sigma \; {}^{(4)}G_{\mu\nu} = -D_\mu K^\mu{}_\sigma + D_\sigma K,$$

$$\gamma^\alpha{}_\mu \gamma^\beta{}_\nu \; {}^{(4)}R_{\alpha\beta} = -\mathcal{L}_n K_{\mu\nu} + R_{\mu\nu} + K K_{\mu\nu} - 2K_\mu{}^\sigma K_{\sigma\nu} - \alpha^{-1} D_\mu D_\nu \alpha.$$

5. Use the results of Problems #1 and #4 to show that the vacuum Einstein equations when expressed in the coordinate system of Problem #1 become:

$$0 = R + K^2 - K_{ij}K^{ij},$$

$$0 = D_i K^i{}_j - D_j K,$$

$$\partial_t \gamma_{ij} - \mathcal{L}_\beta \gamma_{ij} = -2\alpha K_{ij},$$

$$\partial_t K_{ij} - \mathcal{L}_\beta K_{ij} = \alpha R_{ij} + \alpha\big(K K_{ij} - 2K_i{}^k K_{kj}\big) - D_i D_j \alpha,$$

where here the Lie derivative is given by $\mathcal{L}_\beta Q_{ij} = \beta^k D_k Q_{ij} + Q_{kj} D_i \beta^k + Q_{ik} D_j \beta^k$.

6. a) Write a computer program that numerically evaluates the first derivative using finite difference methods of a function $f(x)$ whose values on a fixed grid of points are specified $f(x_i)$, $i = \{0, ..., N-1\}$. Assume that the points x_i are located at $x_i = 2\pi i/N$ on the periodically identified interval $[0, 2\pi)$, and use the standard finite difference expressions for the derivatives:

$$f'(x_0) = [f(x_1) - f(x_{N-1})]/2\Delta x,$$

$$f'(x_i) = [f(x_{i+1}) - f(x_{i-1})]/2\Delta x.$$

for $i = \{1, ..., N-2\}$, and

$$f'(x_{N-1}) = [f(x_0) - f(x_{N-2})]/2\Delta x,$$

where $\Delta x = 2\pi/N$. Demonstrate that your program works by evaluating the derivatives of some known peroidic functions, e.g. $e^{\cos(x)}$, and showing that the difference between your numerical derivatives and the known analytical ones approaches zero as N^{-2}. You might do this by graphing the average error $\epsilon(N) = \sum |f'_{numerical}(x_i) - f'_{analytic}(x_i)|/N$ for several different values of N and then determining whether $\epsilon(N)$ depends on N in the expected way. Turn in a copy of your program (including enough documentation so that we know how your program is supposed to work and what functions you are testing), and graphs demonstrating that your program works.

b) Write a computer program that numerically evaluates the first derivative using pseudo-spectral methods of a function $f(x)$ whose values on a fixed grid of points are specified $f(x_i)$, $i = \{0, ..., N-1\}$, where N is an even integer. Assume that the points x_i are located at $x_i = 2\pi i/N$ on the periodically identified interval $[0, 2\pi)$, and evaluate the derivatives using the expression:

$$f'(x_i) = \sum_{j=0}^{N-1} M_{ij} f(x_j),$$

where

$$M_{ij} = \frac{1}{2}(-1)^{i+j} \cot\left[\frac{\pi(i-j)}{N}\right],$$

for $i \neq j$ and $M_{ij} = 0$ for $i = j$. Demonstrate that your program works by evaluating the derivatives of some known periodic functions, e.g. $e^{\cos(x)}$, and showing that the difference between your numerical derivatives and the known analytical ones approaches zero as $e^{-\lambda N}$ for some $\lambda > 0$. You might do this by graphing the average error $\epsilon(N) = \sum |f'_{numerical}(x_i) - f'_{analytic}(x_i)|/N$ for several different values of N and then determining whether $\epsilon(N)$ depends on N in the expected way. Turn in a copy of your program (including enough documentation so that we know how your program is supposed to work and what functions you are testing), and graphs demonstrating that your program works.