Ph 236c: General Relativity Due 21 May 2006

WEEK 26

Suggested Reading:

1. Pretorius, "Numerical Relativity Using a Generalized Harmonic Decomposition," *Class. Quant. Grav.* **22**, 425–452 (2005); gr-qc/0407110.
2. Lindblom, et al. "A New Generalized Harmonic Evolution System," *Class. Quant. Grav.* **23**, S447-S462 (2006); gr-qc/0512093.
3. Pretorius, "Simulation of Binary Black Hole Spacetimes with a Harmonic Evolution Scheme," *Class. Quant. Grav.* **23**, S529-S552 (2006); gr-qc/0602115.

Problems

1. a) Let ψ_{ab} denote the 4-dimensional metric tensor. Show that the principal parts of the 4-dimensional Ricci tensor (i.e. the terms containing second derivatives of ψ_{ab}) can be written in the form

$$R_{ab} \simeq -\frac{1}{2}\psi^{cd}\partial_c\partial_d\psi_{ab} + \nabla_{(a}\Gamma_{b)},$$

where $\Gamma_a = \psi^{bc}\Gamma_{abc}$, Γ_{abc} is the Christoffel symbol, and $\nabla_a\Gamma_b = \partial_a\Gamma_b - \psi^{cd}\Gamma_{cab}\Gamma_d$.
b) Show that the covariant scalar wave operator acting on the coordinate function x^b can be written as

$$\psi_{ab}\nabla^c\nabla_c x^b = -\Gamma_a.$$

2. a) Let $H_a(x, \psi)$ be a prescribed function of the coordinates x^a and the 4-metric ψ_{ab}, e.g. $H_a(x, \psi) = \psi_{ab}H^b(x)$. Assume that the coordinates are fixed by requiring that the scalar wave operator acting on the coordinate functions give H_a, i.e. that $0 = H_a + \Gamma_a$. When this choice of coordinates is made, show that the vacuum Einstein equations can be written as

$$0 = R_{ab}(\psi) - \nabla_{(a}\mathcal{C}_{b)},$$

where $\mathcal{C}_a = H_a + \Gamma_a$. The quantity $\mathcal{C}_a$ functions like a constraint, with $\mathcal{C}_a = 0$ being the condition that the coordinates satisfy the scalar wave equation with the prescribed source function H_a.
b) Show that the principal parts (i.e. the parts containing second derivatives of the metric) of this form of the vacuum Einstein equations are given by

$$R_{ab}(\psi) - \nabla_{(a}\mathcal{C}_{b)} \simeq -\frac{1}{2}\psi^{cd}\partial_c\partial_d\psi_{ab}.$$

3. a) Use the Bianchi identity to infer that the constraints satisfy the vector wave equation:

$$\nabla^c\nabla_c\mathcal{C}_a = -R_{ab}\mathcal{C}^b = -\mathcal{C}^b\nabla_{(a}\mathcal{C}_{b)}.$$

b) Show that if $\mathcal{C}_a = 0$ on a spacelike surface, and if the Hamiltonian and momentum constraints are satisfied on this surface, then $\partial_t\mathcal{C}_a = 0$ on this surface as well.

4. Modify the generalized harmonic evolution system by adding the following multiples of the constraints to the equations,

$$0 = R_{ab} - \nabla_{(a}\mathcal{C}_{b)} + \gamma_0\Big[t_{(a}\mathcal{C}_{b)} - \frac{1}{2}\psi_{ab}t^c\mathcal{C}_c\Big],$$

where $\mathcal{C}_a = H_a + \Gamma_a$ is the generalized harmonic constraint, t_a is a future directed timelike vector, and γ_0 is a positive constant.
a) Show that the resulting evolution system is hyperbolic.
b) Use the Bianchi identities to show that the constraints for this system satisfy the equation,

$$0 = \nabla^c\nabla_c\mathcal{C}_a - 2\gamma_0\nabla^b\Big[t_{(b}\mathcal{C}_{a)}\Big] + \mathcal{C}^c\nabla_{(a}\mathcal{C}_{b)} - \frac{1}{2}\gamma_0 t_a\mathcal{C}^c\mathcal{C}_c.$$

5. Write a finite difference code to evolve the 1D scalar wave system,

$$\begin{aligned}\partial_t\psi &= -\Pi,\\ \partial_t\Pi &= -\partial_x\Phi_x,\\ \partial_t\Phi_x &= -\partial_x\Pi\end{aligned}$$

on the computational domain $0 \le x < 2\pi$, with the points $x = 0$ and $x = 2\pi$ periodically identified. Evolve initial data corresponding to some known analytical solution to these equations, e.g. $\psi(x,t) = \alpha e^{\cos(t-x)} + (1-\alpha)e^{\cos(t+x)}$ for some values of the constant α. Do convergence tests to verify that your numerical solutions converge to the known analytical solutions.

6. Repeat Problem #5 using spectral methods.