WEEK 3: CONSERVATION LAWS, STRESS-ENERGY TENSOR, ACCELERATED OBSERVERS, CONNECTION AND GEODESICS

Revision 1 - With Supplementary Reading added

Recommended Reading:

1. MTW: Chapter 5; Also Sec. 4.7.
2. Blandford and Thorne chap. 0601.2.K: Secs. 1.11 and 1.12. **Be sure to use version 0601.2.K of chapter 1 of Blandford and Thorne at**
   http://www.pma.caltech.edu/Courses/ph136/yr2006/text.html
3. MTW Chapter 6 (especially Sec. 6.6)
4. MTW Sections 8.1–8.5
5. MTW Section 9.6 [on commutators]
6. Blandford and Thorne Sec. 23.3 (of version 0623.1.K of Chap. 23).

Possible Supplementary Reading:

Conservation Laws and Stress-Energy Tensor

7. Schutz (*A First Course in General Relativity*): Chapter 4. This discusses the stress-energy tensor from a more elementary viewpoint than MTW or Blandford and Thorne, with the central focus being on a perfect fluid and on analyses in an inertial reference frame rather than frame-independent.
8. Wald (*General Relativity*): Section 4.2. This discusses the stress-energy tensor and the flow of 4-momentum through spacetime in much the same language as mine.
9. Wald: Appendix B. This discusses integration over spacetime (and other manifolds) and over 3-surfaces in spacetime (and submanifolds embedded in other manifolds), using the language of exterior calculus.
10. Carroll (*Spacetime and Geometry*): Sections 1.9 and 1.10. This discusses the stress-energy tensor, including its connection to field theory (not treated in our course).
11. Hartle (*Gravity*: Sections 22.1, 22.2. This is a clear, elementary discussion of the stress-energy tensor and conservation of 4-momentum.

Accelerated Observers

[The flat-spacetime metric in the coordinates of a family of uniformly accelerated observers is called the *Rindler Metric*. I should have told you that in class.]

12. Wald, pp. 149–152. This is a moderately sophisticated treatment of the Rindler metric
13. Carroll, pp. 403–405. So is this.

Connection Coefficients and geodesics

14. Schutz (*A First Course in General Relativity*): Chapter 5. This covers essentially the same material as the recommended sections of MTW Chapter 8, but in a more elementary and leisurely way.
15. MTW, chapters10, This covers parallel transport and connection coefficients in great detail, in manifolds where you might not have a metric.
16. Wald (*General Relativity*): Section 3.1. This covers connection coefficients in a quick, mathematically sophisticated way.

17. Carroll: Sections 3.1–3.5. This is a very nice treatment of connection coefficients and geodesics.
18. Hartle: Sections 8.1–8.3. So is this.

Problems. Each problem is worth 10 points. The total problem set is worth a maximum of 50 points. Your solutions must be turned in at the beginning of class next Wednesday, 25 October.

1. **Global conservation of 4-momentum in a Lorentz frame**. Exercise 1.26 of Blandford and Thorne.

2. **Rest-mass-flux 4-vector, Lorentz contraction of rest-mass density, and rest-mass conservation for a fluid.** Exercise 1.24 of Blandford and Thorne.
3. **Stress-energy tensor and energy-momentum conservation for a perfect fluid**. Exercise 1.27 of Blandford and Thorne: Parts (b) and (c). Do not do part (a) unless you feel a little shaky about the solution to this that I gave in my Monday lecture. Note that the notation here differs from my lecture: $P^{\alpha\beta}$ is the Blandford-Thorne notation for the tensor that projects orthogonal to an observer's world line (the observer's spatial metric); in my lecture I used the notation $\gamma^{\alpha\beta}$.

4. **Inertial mass per unit volume**. Exercise 5.4 of MTW.

5. **Entropy and the Second Law of Thermodynamics**
It turns out, as we shall see later in this course, that the entropy of a system that has many microscopic degrees of freedome (e.g. a gas) can be described macroscopically by an *entropy density-flux 4-vector* $\vec{s}$.
   a. Give a definition of this $\vec{s}$ analogous to our definition of the charge-current 4-vector $\vec{J}$ and the stress-energy tensor $\mathbf{T}$.
   b. Relying on that definition, formulate the second law of thermodynamics in a geometric, frame-independent manner, and discuss the justification for your formulation.

6. **Stress-Energy Tensor for the Electromagnetic Field**. Exercise 5.1 of MTW. Note: I encourage you to read carefully the discussion of energy-momentum conservation for the electromagnetic field interacting with a block of electrically charged rubber in Sec. 5.10 of MTW.

7. **Charge-Current 4-Vector for a Point Particle**
I claim that the charge-current 4-vector for a classical point particle with charge q, proper time τ, world line $\mathcal{P}(\tau)$, and 4-velocity $\vec{u}(\tau) = d\mathcal{P}/d\tau$ is

$$\vec{J}(\mathcal{P}') = q \int \vec{u}(\tau)\delta_4[\mathcal{P}(\tau), \mathcal{P}']d\tau \; , \tag{2}$$

where $\delta_4[\mathcal{P}, \mathcal{P}']$ is the 4-dimensional Dirac delta function whose spacetime integral over $\mathcal{P}'$, $\int \delta_4[\mathcal{P}, \mathcal{P}']d^4\Omega'$, is unity if the region of integration includes the point $\mathcal{P}$ and zero otherwise.

Verify that expression (2) is, indeed, the particle's charge-current 4-vector by showing that it satisfies the defining property of $\vec{J}$: $\int_{\mathcal{S}} J^\alpha d^3\Sigma_\alpha = q$ if the particle passes through the 3-volume $\mathcal{S}$ in the positive direction, and $\int_{\mathcal{S}} J^\alpha d^3\Sigma_\alpha = 0$ if the particle does not pass through $\mathcal{S}$. Note: In your calculation you may wish to introduce a local Lorentz reference frame in which, at the event where the particle passes through $\mathcal{S}$, $\mathcal{S}$ lies in a coordinate plane—the plane $t = 0$ if $\mathcal{S}$ is spacelike; the plane $z = 0$ if $\mathcal{S}$ is timelike.

8. **Stress-Energy Tensor for a point particle.**
   a. Consider a point particle with rest mass m, proper time τ, world line $\mathcal{P}(\tau)$, and 4-velocity $\vec{u} = d\mathcal{P}/d\tau$. Write down a frame-independent equation for the stress-energy tensor of this particle $\mathbf{T}(\mathcal{P})$ as an integral along the particle's world line analogous to Eq. (2) for the charge-current 4-vector of a point particle.
   b. Verify that this is indeed the particle's stress-energy tensor by performing a surface integral analogous to that used for the charge-current 4-vector in the previous problem.
   c. Your expression in part (a) only works for a particle with finite rest mass. Construct an alternative expression for the stress-energy tensor involving an "affine parameter" along the particle's world line $\mathcal{P}(\zeta)$, such that the particle's 4-momentum is $\vec{p} = d\mathcal{P}/d\zeta$. More specifically, write $\mathbf{T}(\mathcal{P})$ as an integral along the world line that involves $\vec{p}(\zeta)$, $\mathcal{P}(\zeta)$, and not in any way $m(\zeta)$. Your expression should work for particles with finite rest mass and also for particles with zero rest mass.

9. 4-Acceleration in uniformly accelerated reference frame
Exercise 6.7 of MTW. It is the magnitude of the 4-acceleration, $\vec{a} = \nabla_{\vec{u}}\vec{u}$, that you are asked to compute. Do this computation in two different coordinate frames: (a) In the inertial frame, and (b) in the accelerated coordinate frame using the connection coefficients associated with the metric (6.18)

10. Formula for Connection Coefficients in an Arbitrary Basis
Read MTW Exercise 8.14; then do MTW Exercise 8.15.

11. A Sheet of Paper in Polar Coordintes Exercise 8.5 of MTW. Augment part (d) by the following: By drawing pictures (as I did in class), convince yourself that your answer for your connection coefficients in the noncoordinate basis are correct.

11. Some Useful Formulas in Coordinate Frames
MTW Exercise 8.16 parts (a), (c), (d) but not (b). *Note:* This exercise is rather important; we shall need the formulas it derives.