WEEK 4: SPACETIME CURVATURE, LAWS OF PHYSICS IN CURVED SPACETIME, AND EINSTEIN FIELD EQUATION

Point Values for Problems Have Been Changed as of 29 October; no other changes

Recommended Reading:

1. MTW:
   a. Secs. 8.6 and 8.7
   b. Secs. 16.2 and 16.3
   c. Secs. 17.1 — 17.4 and 17.6
2. Blandford and Thorne [at http://www.pma.caltech.edu/Courses/ph136/yr2006/]: Secs. 24.2–24.8 and 24.9.1 (of chapter 24).

Possible Supplementary Reading:

3. MTW Chapters 11 and 13 (but not Sec. 13.6 which we will study later), on Geodesic Deviation and Spacetime Curvature. This is an in-depth treatment of these topics, initially (Chap. 11) without a metric and then (Chap. 13) with a metric.
4. MTW: The remainder of Chapters 16 and 17 (portions not contained in items 1.b and 1.c).
5. Hartle (*Gravity*), Chaps. 6, 7, 21 and 22. Chapters 6 and 7 are a leisurely, elementary introduction; 21 and 22 add the mathematical detail.
5. Schutz (*A First Course in General Relativity*): Chapters 6, 7 and 8. This is an elemetary-to-intermediate-level treatment.
6. Carroll (*Spacetime and Geometry*), Sections 3.6–3.11 and Chapter 4. This is an intermediate-level treatment, some of it from a different point of view than my own, and with some emphasis on the connections to field theory and to some contemporary topics.
7. Wald (*General Relativity*): Chaps. 3 and 4, but not Sec. 4.3b. This is a concise, mathematically sophisticated treatment.
8. MTW Chapter 7. This describes what happens when one tries to construct a relativistic theory of gravity as a linear field theory in flat spacetime, and why such theories don't work. For a sophisticated follow up on this chapter's tensor theory of gravity in flat spacetime, see Section 5 of Box 17.2 — which describes, in brief, how the attempt to make the tensor theory self consistent leads to general relativity.
9. Richard P. Feynman, Fernando B. Morinigo, and William G. Wagner, *Feynman Lectures on Gravitation* (Addison Wesley, 1995). Pages x–xvi of the forward to this book (by Preskill and Thorne) outline Feynman's approach to the derivation of the Einstein field equation, an approach closely related to that sketched in item 4 above. Chapters 1–6 of this book present Feynman's derivation.

Problems. All problems are worth **15** points unless otherwise indicated. The maximum number of points that will be given to anyone on this set is 50.

1. **Formula for Components of the Riemann tensor in an Arbitrary Basis**
   a. Blandford and Thorne, Exercise 24.8
   b. If you prefer a more sophisticated route to this result: Read MTW Exercise 11.3; then do MTW Exercise 11.4.
2. **"Flat" Friedman Universe: Local Lorentz Frame and Curvature**
   The spacetime metric for a flat Friedman Universe is

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) = -dt^2 + a^2(t)\delta_{jk}dx^j dx^k \ . \tag{2}$$

   a. Verify that a particle at rest in the (t, x, y, z) coordinate system [i.e., a particle with world line (x, y, z) = constant] moves along a geodesic of spacetime, and thus is freely falling.
   b. Consider a freely falling particle that is at rest at the spatial origin, $(x, y, z) = (0, 0, 0)$. In the vicinity of this particle's world line introduce new coordinates

$$T = t + \frac{1}{2}\frac{\dot{a}}{a}a^2(x^2 + y^2 + z^2) \ , \quad X = ax, \ Y = ay, \ Z = az \ , \tag{3}$$

   where the dot means d/dt. Show that these new (T, X, Y, Z) coordinates are a local Lorentz frame of the freely falling particle — i.e., show that in this coordinate system, up to corrections that are second order in spatial distance from the spatial origin, $g_{\mu\nu} = \eta_{\mu\nu}$.
   c. The freely falling observer at $(x, y, z) = (0, 0, 0)$ measures the motion of a nearby freely falling particle whose world line in the original coordinate system is $(x, y, z) = (\delta x, \delta y, \delta z)$. The observer studies the particle in the observer's local Lorentz coordinate system $\{x^{\hat{\alpha}}\} \equiv (T, X, Y, Z)$. By comparing the particle's motion with the equation of geodesic deviation in the local Lorentz frame, show that the tide-producing local Lorentz components of the Riemann tensor are

$$R^{\hat{j}}{}_{\hat{0}\hat{k}\hat{0}} = -\frac{\ddot{a}}{a}\delta^j_k \ . \tag{4}$$

   d. There are also nonzero purely spatial components of Riemann, $R^{\hat{j}}{}_{\hat{k}\hat{l}\hat{m}}$, which cannot be computed in the above manner. Use symmetry arguments to show that they must have the form

$$R_{\hat{j}\hat{k}\hat{l}\hat{m}} = \mathcal{R}(t)(\delta_{\hat{j}\hat{l}}\delta_{\hat{k}\hat{m}} - \delta_{\hat{j}\hat{m}}\delta_{\hat{k}\hat{l}}) \ , \tag{5}$$

   where $\mathcal{R}(t)$ is some function of t. The remainder of this exercise is directed toward verifying Eq. (4) and computing the $\mathcal{R}(t)$ of Eq. (5).
   d. Compute the connection coefficients in the (t, x, y, z) coordinate basis, either by hand or using tensor-manipulation software.†

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† For details on where to find the relevant computer software, see our course home page, http://www.pma.caltech.edu/~ph236/yr2007/] . I especially recommend the Mathematica programs associated with Hartle's textbook (see the link on our home page). They are easy to understand, and with them as templates, you can write other tensor manipulation software.

e. Compute the components of the Riemann tensor in the (t, x, y, z) coordinate basis, either by hand or using tensor-manipulation software.

f. Transform the components of Riemann to the local Lorentz frame of the freely falling observer. Thereby verify that the only nozero components are those of Eqs. (4) and (5), and deduce the value of $\mathcal{R}(t)$.

3. Geometrized Units. [5 Points]

a. Compute the values of the following quantities in geometrized units (units with $G = c = 1$); express your answers in centimeters or some power thereof: Planck's constant, $\hbar$; the charge of the electron, e; the fine structure constant, $e^2/(\hbar c)$; your mass; the mass of the sun; the mass of the Earth.

b. Use dimensional considerations to restore the factors of G and c in all the numbered equations in this problem set (below). You might want to wait and do this after you have studied all of the problems that contain numbered equations.

4. Precession of the Equinoxes.

As an exploration of curvature coupling effects in general relativity, do MTW Exercise 16.4. [*Note:* intrinsic angular momentum is discussed, in any Lorentz frame (global or local), in Box 5.6.]

5. Quantum-Gravity-Induced Curvature Coupling in Maxwell's Equations [Problem due to Walter Goldberger with modifications by Kip] [NOTE: In my lecture on Wednesday I forgot about the possibility of the scalar curvature R generating curvature coupling, so my argument that the Maxwell equations cannot have curvature coupling was wrong. Here is an important counter example. - Kip]

In Box 16.1 of MTW it is argued (via the equivalence principle) that, because the electromagnetic field tensor is physically measureable and the Maxwell equations expressed in terms of it involve only first derivates, not second, there is unlikely to be any curvature coupling in those Maxwell equations; and, in particular, the Maxwell equations should read

$$\nabla_\nu F^{\mu\nu} = 4\pi J^\mu \ , \quad \nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 0 \qquad (WG.1)$$

in curved spacetime, as in flat. (Here the notation is $\nabla_\nu F^{\mu\nu} \equiv F^{\mu\nu}{}_{;\nu}$.)

a. Show that conservation of charge follows from the first of these equations; that is, show that

$$\nabla_\mu J^\mu = 0.$$

b. As will be explained below, quantum gravity might induce a curvature coupling of the following form:

$$\nabla_\nu[(1 + \alpha R)F^{\mu\nu}] = 4\pi J^\mu, \quad \nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 0 \ ; \qquad (WG.2)$$

here α is some constant and $R = R^\mu{}_\mu$ is the scalar curvature of spacetime. Like Eq. (WG.1), these reduce to the familiar Maxwell equations in flat spacetime (since $R = 0$ there). The fact that the second of these equations is unmodified means that we can still write $F_{\mu\nu}$ in terms of a vector potential, $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$, and the theory is still gauge invariant. Show that this version of the Maxwell equations, like the more conventional version, implies charge conservation. [*Hint:* first show that antisymmetry of $\mathbf{F}$ implies $\nabla_\mu \nabla_\nu [(1+\alpha R)F^{\mu\nu}] = (1+\alpha R)\nabla_\mu \nabla_\nu F^{\mu\nu}$, then use that antisymmetry to show that this expression entails commutation of covariant derivatives, and show that the curvature terms produced by that commutation give a vanishing result.]

c. Quantum gravity is known to introduce curvature couplings into the laws of physics, with coupling constants that involve the Planck length, $l_P \equiv \sqrt{\hbar}$. Reexpress this Planck length in cgs units by restoring the appropriate factors of G and c, and evaluate it numerically (you should get an extremely small number). By dimensional considerations, estimate the numerical value of the coupling constant α which quantum gravity might induce.

d. Perform a 3+1 split of these modified Maxwell equations in the local Lorentz frame of some observer—e.g., an experimenter in the space shuttle; i.e., rewrite them in terms of the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$, and the charge and current densities ρ and $\mathbf{j}$ that the observer measures. Discuss the physical manifestations of the curvature coupling that these 3+1 equations predict; pay attention to the fact that R vanishes in vacuum, and that at the surface of some solid body $\boldsymbol{\nabla} R$ will have a delta-function behavior. You might want to consider, for example, Gauss's law which usually expresses the total charge inside a body as a surface integral of the electric field sticking out of the body. Estimate the dimensionless magnitude of these physical effects. [*Answer:* They are so tiny that there is no hope at all to measure them with forseeable technology.]

6. Newtonian Limit of General Relativity.

[*Note:* This problem should be easier than some of the preceding ones. It will help you get insight into the relationship between general relativity and Newton's description of gravity.]

a. MTW Exercise 17.6

b. MTW Exercise 16.1. *Note:* In the spacetime metric (16.2a) of this exercise we see two places that the Newtonian potential Φ enters: the time-time part of the metric, $g_{00} = -1-2\Phi$ (the "correspondence relation" dealt with in Exercise 17.6), and the space-space part of the metric, $g_{jk} = (1+2\Phi)\delta_{jk}$. *Show* that only the time-time (correspondence-relation) Φ term influences the motion of the fluid. Show this by prepending some coefficient γ to the space-space part of the metric, i.e. by writing the metric as

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\gamma\Phi)(dx^2+dy^2+dz^2) \ , \qquad (4)$$

and by then demonstrating that γ does not enter into the fluid's equation of motion. As we shall see later, the Φ term in the space-space part of the metric is actually a relativistic "*post*-Newtonian" correction to the Newtonian theory of gravity.

7. Stress Tensor for Newtonian Gravitational Field.
[*Note:* This problem is also relatively easy; it is in part an exercise in index manipulation, and in part a vehicle for understanding Newtonian gravity more deeply.]

Define a stress tensor for the Newtonian gravitational field as follows:

$$T_{\mathrm{N}\ jk} = \frac{1}{4\pi}\left(\Phi_{,j}\Phi_{,k} - \frac{1}{2}g_{jk}\Phi_{,l}\Phi^{,l}\right) . \tag{5}$$

This stress tensor lives in the ordinary, flat, Euclidean space of Newtonian physics.

a. Explain why, if expression (5) is truly a stress tensor for the Newtonian gravitational field, the Newtonian equation of motion for a perfect fluid [the second of Eqs. (16.3a) of MTW] should be reexpressible as

$$\rho\frac{dv_j}{dt} = -(T_{\mathrm{N}\ jk} + pg_{jk})_{,k} \tag{6}$$

in a Cartesian coordinate system. Verify that this equation is, indeed, equivalent to (16.3a) of MTW.

b. Explain why you would expect the relativistic law of momentum conservation to reduce to the following in the Newtonian limit:

$$(\rho v_j)_{,t} + (T_{\mathrm{N}\ jk} + pg_{jk} + \rho v_j v_k)_{,k} = 0 . \tag{7}$$

Explain the physical meaning of each of the terms in this equation. Verify that this equation is, in fact, just a rewritten form of Eq. (6).

c. For further insight into a full stress-energy tensor for gravity in the Newtonian limit, see Box 12.23 of Chapter 12 (version 0612.2.K.pdf) of Blandford and Thorne.

8. Nordström's Theory of Gravity [5 Points]
MTW Exercise 17.8

9. A Curved-Spacetime Variant of the Newtonian Theory of Gravity, and Prior Geometry [10 Points]
Consider the following theory of gravity: Spacetime is endowed with a metric $\mathbf{N}(\_\,,\_)$ that is flat, i.e. whose Riemann tensor vanishes. It is also endowed with a vector field $\vec{w}$ that is timelike, has unit norm ($\mathbf{N}(\vec{w},\vec{w}) = -1$), and has vanishing gradient ($^{(N)}\boldsymbol{\nabla}\vec{w} = 0$ where $^{(N)}\boldsymbol{\nabla}$ is the gradient defined using the parallel transport law embodied in the flat metric $\mathbf{N}$). There exists a scalar gravitational field that is generated by the trace of the stress-energy tensor according to the law (in slot-naming index notation)

$$(N^{\alpha\beta} + w^\alpha w^\beta)\Phi_{|\alpha\beta} = -4\pi G N_{\alpha\beta}T^{\alpha\beta} .$$

Here the subscript | indicates the gradient $^{(N)}\boldsymbol{\nabla}$ and G is Newton's gravitational constant. This scalar field and the flat metric are combined together in the following way to produce a tensor field $\mathbf{g}$:

$$\mathbf{g} = \mathbf{N} - 2\Phi\vec{w}\otimes\vec{w} .$$

In all of the above mathematics the role of the metric is played by $\mathbf{N}$. However, once $\mathbf{g}$ is determined, $\mathbf{g}$ then plays the role of a curved spacetime metric; and all of the familiar laws of physics (Maxwell's equations, etc.) take their standard forms in that curved spacetime. For example, freely falling particles move along geodesics of $\mathbf{g}$.

a. Show that there exists a coordinate system in which the components of $\mathbf{N}$ and $\vec{w}$ are $N_{\alpha\beta} = \eta_{\alpha\beta}$ (the standard Minkowski metric components of flat spacetime) and $w^0 = 1$, $w^j = 0$.

b. Show that in this coordinate system, the scalar field Φ satisfies the standard Newtonian gravitational law

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}\right)\Phi = 4\pi G(T^{00} - T^{xx} - T^{yy} - T^{zz}) \ .$$

This is Newton's gravitational source equation (the Laplace equation; "action at a distance" law) with the flat-space trace of the stress-energy tensor producing a Newtonian gravitational field Φ.

c. Show that in this coordinate system, the curved-spacetime metric $\mathbf{g}$ takes the form

$$ds^2 = -(1 + 2\Phi)dt^2 + dx^2 + dy^2 + dz^2 \ .$$

d. Show that geodesic motion for test particles in this metric, in nearly Newtonian situations (velocities small compared to light and $|\Phi| \ll 1$) reduces, in this coordinate system, to the standard Newtonian equation of motion $dx^j/dt = -\partial\Phi/\partial x^j$.

e. Thus, this theory is identical to Newton's theory in nearly Newtonian situations, and even in highly relativistic situations it is still an action-at-a-distance theory with instantaneous propagation of the gravitational force. Nevertheless, this theory is in accord with the Principle of Relativity: All the laws are expressed in frame-independent, geometric language. This is another example demonstrating that the Principle of Relativity plus the Newtonian limit do not lead, necessarily, to Einstein's General Relativity Theory. Here, as in Nordstrom's theory (Sec. 17.6 of MTW), the key to arriving at a theory radically different from General Relativity is *Prior Geometry.* Show that the tensor field $\mathbf{N}$ and vector field $\vec{w}$ appearing in this theory are, indeed, prior geometric objects in the sense of Sec. 17.6 of MTW, and explain why Φ is an *auxiliary gravitational field.* General relativity, by contrast with this Newton-like theory, has no prior geometry and no auxiliary gravitational fields.