Ph 236a: General Relativity 31 October 2007

WEEK 5: RELATIVISTIC STARS — STATIC AND SLOWLY ROTATING; LOCAL INERTIAL FRAMES AND PROPER REFERENCE FRAMES; SYMMETRIES AND CONSERVATION LAWS

Recommended Reading:

1. Relativistic Stars:
   a. MTW Chapter 23 plus sections 24.1 and 24.2, and Exercise 26.1.
   b. Blandford and Thorne, sections 25.1 and 25.3 of version 0625.1.K.pdf of chapter 25, available at http://www.pma.caltech.edu/Courses/ph136/yr2006/ .
   [For a solution to this exercise, see Sections III and IV of the paper "Slowly Rotating Relativistic Stars. I. Equations of Structure" by James B. Hartle, *The Astrophysical Journa*, **150**, 1005–1029 (1967). I do not know of any textbook treatment of this, though such a treatment should exist.]
2. Local Inertial Frames and Proper Reference Frames
   a. MTW Section 13.6
   b. Section 23.5 of chapter 23 (Version 0623.1.K.pdf) of Blandford and Thorne. This is a more elementary treatment, and from a more familiar, less sophisticated viewpoint.
3. Symmetries and Conservation Laws
   a. MTW Secs. 25.2, 25.3, 25.4, and Box 19.1.
   [Note: Box 19.1 gives a more general discussion of the conservation law for the source's angular momentum than I developed in my Wednesday lecture. But the basic idea is the same as in my Wednesday lecture. Box 19.1 omits the key idea that the conservation laws for the source's mass M and angular momentum $\vec{S}$ (or J in my lectures) owe their existence to the symmetries of spacetime at radial infinity.]

Possible Supplementary Reading:

4. Relativistic Stars
   a. Schutz (*A First Course in General Relativity*): Chapter 10. This covers the same material as MTW Chapter 23.
   c. Wald (*General Relativity*): Sections 6.1 and 6.2. This covers some of that same material, but in a quicker and more mathematically sophisticated way.
   b. Carroll *Spacetime and Geometry*, pp. 230–236.
   c. Hartle *Gravity*: Chap. 24. This discusses the construction of relativistic stellar models.
   d.. For a detailed discussion of neutron stars: Chapter 9 of Stuart L. Shapiro and Saul A. Teukolsky, *Black Holes, White Dwarfs, and Neutron Stars* (Wiley, New York, 1983).
4. Local Inertial Frames and Proper Reference Frames:
   a. Regretably, most textbooks omit any discussion of local Lorentz frames or proper reference frames along the world line of some observer. The closest they come is to construct local inertial frames in the vicinity of a single event in spacetime;

these are often called *Riemann normal coordinates*. See, e.g., Carroll, pp. 111–113 and Wald p. 42.

   b. Hartle's undergraduate textbook is an exception. It does discuss local Lorentz frames all along the world line of a freely falling observer, and calls them "freely falling frames". Another, technical term is "Fermi Normal Coordinates." See Hartle Sec. 8.4.

5. Symmetries and Conservation Laws:
   a. Hartle, Sec. 8.2. This is an elementary discussion.
   b. Wald, Sec. C.3 (pp. 441-444). This is a sophisticated discussion. It includes a brief introduction to Lie derivatives.
   c. Carroll, Sec. 3.8 on Killing vectors and associated conservation laws; Appendix B on Lie derivatives.

Note Concerning Computer-Aided Tensor Analysis: Some of the exercises involve calculations of curvature tensors that would be long and tedious if carried out by hand. I suggest you use tensor-manipulation computer software. Sources for such software are given on our course home page, http://www.pma.caltech.edu/~ph236/yr2007/] . I especially recommend the Mathematica programs associated with Hartle's textbook; see the link on our home page). They are easy to understand, and with them as templates, you can write other tensor manipulation software problem set in which you are asked to do computer-aided tensor calculations. There are links to other relevant software packages on the GRTensor web site: http://grtensor.phy.queensu.ca/

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Problems

Note: The point values of each problem is shown after its title. As usual, the maximum number of points that will be given for this set is 50.

Problems Related to My Material on Relativistic Stars

1. **Flat Friedman Universe.** [15 points]
   This exercise is designed to give you practice at building a model for a physical system in the same way as I did in my lecture on Monday. I dealt with a static, spherical star. You will deal with a simple model for our universe and its evolution.

   Idealize our universe as being spatially homogeneous and Euclidean. In other words, through every event there passes a 3-dimensional spacelike hypersurface that has flat, Euclidean geometry. Consider any specific such surface; call it $\mathcal{S}$. Consider the family of observers on $\mathcal{S}$, who see $\mathcal{S}$ as their 3-spaces of simultaneity. Call them *homogeneous observers*. When these homogeneous observers make measurements of their surroundings, they all see identically the same things—the same density, the same pressure, the same temperature, the same spacetime curvature, etc.

a. Use these physical features of the universe to construct coordinates $\{t, x, y, z\}$ in which the spacetime metric takes the flat Friedman form

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2) \ . \tag{2}$$

Explain in detail, step-by-step, how the coordinates are constructed, and show that your construction leads to a metric of the form (2).

b. What are the basis vectors (orthonormal tetrad) $\vec{e}_{\hat{\alpha}}$ of each homogeneous observer's proper reference frame? Express them in terms of the coordinate basis vectors.

c. This universe is filled with perfect fluid, that is at rest with respect to the homogeneous observers. What are the components of the fluid's stress-energy tensor in these observer's proper reference frames?

d. These observers apply their laws of local energy conservation and local momentum conservation to this fluid. What do those laws say about the evolution of the fluid's total density of mass-energy ρ and pressure P, in terms of the evolution of $a(t)$.

e. Set $H^{\hat{\alpha}\hat{\beta}} = G^{\hat{\alpha}\hat{\beta}} - 8\pi T^{\hat{\alpha}\hat{\beta}}$, where $\mathbf{G}$ is the Einstein tensor and $\mathbf{T}$ the stress-energy tensor. Energy-momentum conservation and the contracted Bianchi identities guarantee that $\vec{\nabla} \cdot \mathbf{H} = 0$ before the Einstein equations have been imposed. Show that this, plus the homogeneity imply that if we impose just one of the 10 Einstein equations $H^{\hat{0}\hat{0}} = 0$, all the other Einstein equations are guaranteed to be satisfied.

f. Compute $G^{\hat{0}\hat{0}}$ (using computer software — e.g., if using Hartle's software, by computing the coordinate components $G_{00} = 0$ and then transforming to the orhonormal basis.

g. Show that the Einstein equation gives a relationship between the time derivative of $a(t)$ and the universe's density $\rho(t)$.

h. Explain why this relation and the one deduced in part (d) and an equation of state $P = P(\rho)$ are sufficient to predict the universe's evolution, once initial conditions have been given.

2. Einstein Tensor for Relativistic Star [5 points]
For the relativistic star with spacetime metric

$$ds^2 = -e^{2\Phi}dt^2 + \frac{dr^2}{1 - 2m/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ , \tag{6}$$

[where Φ and m are functions of r], use a computer to calculate the components of the Einstein tensor in the fluid's proper reference frame

$$\vec{e}_{\hat{0}} = e^{-\Phi}\frac{\partial \mathcal{P}}{\partial t} \ , \quad \vec{e}_{\hat{r}} = \sqrt{1 - 2m/r}\frac{\partial \mathcal{P}}{\partial r} \ , \quad \vec{e}_{\hat{\theta}} = \frac{1}{r}\frac{\partial \mathcal{P}}{\partial \theta} \ , \quad \vec{e}_{\hat{\phi}} = \frac{1}{r\sin\theta}\frac{\partial \mathcal{P}}{\partial \phi} \ . \tag{7}$$

Note: You can do this in either of two ways: perform the computation directly in the (non-coordinate) orthonormal frame of Eq. (4); or perform it in the corresponding coordinate frame, and then transform to the orthonormal frame (4). Your answer should agree with the one discussed in Chapter 23 of MTW (pages 602 to 606).

3. **Schwarzschild Geometry in Isotropic Coordinates.** [5 points]
Show, either by hand or using a computer, that if one changes radial coordinates from r to r' via the transformation

$$r = r'(1 + M/2r')^2 \ , \tag{10}$$

the Schwarzschild metric (9) takes on the new form

$$ds^2 = -\left(\frac{1 - M/2r'}{1 + M/2r'}\right)^2 dt^2 + \left(1 + \frac{M}{2r'}\right)^4 [dr'^2 + r'^2(d\theta^2 + \sin^2\theta d\phi^2)] \ . \tag{11}$$

This new (t, r', θ, ϕ) coordinate system is called "isotropic" because in it the spatial part of the metric is written as a function of r' times the flat, 3-dimensional Euclidean metric (in spherical coordinates), and this Euclidean metric does not pick out any direction as special.

4. **Embedding Diagram for a Wormhole.** [10 points]
   a. Show that in the isotropic coordinates of Eq. (11), the radial coordinate r' is everywhere spatial—i.e., $\partial\mathcal{P}/\partial r'$ points in a spacelike direction. This contrasts with Schwarzschild's original radial coordinate r, which is a spatial coordinate at $r > 2M$ but is timelike at $r < 2M$.
   b. Consider an equatorial two-dimensional surface t =constant, $\theta = \pi/2$ =constant. Construct an embedding diagram for this surface [analogous to the embedding diagram, Fig. 23.1, for an equatorial surface in the spacetime of a static, spherical star]. To do this construction, use the isotropic coordinates of Eq. (11); i.e., in a flat Euclidean space with coordinates $(\bar{r}, \bar{z}, \bar{\phi})$ and $ds^2 = d\bar{r}^2 + d\bar{z}^2 + \bar{r}^2 d\bar{\phi}^2$, construct a 2-surface labeled by surface coordinates (r', ϕ) with the same 2-metric as you read off of Schwarzschild, Eq. (11). Your embedding diagram should turn out to be a "wormhole" (also called an "Einstein-Rosen bridge") connecting two asymptotically flat spaces; cf. the pictures on page 837 of MTW and the first few pages of Chapter 14 of my book *Black Holes & Time Warps: Einstein's Outrageous Legacy*.
   c. Derive an equation $\bar{z} = \bar{z}(\bar{r})$ for the shape of this surface. Your answer should be identical to Eq. (23.34b) of MTW. Explain why it had to be so.

5. **Physical Measurements in the Spacetime of a Slowly Rotating Star** [10 points]
The spacetime metric inside a slowly rotating star has the form

$$ds^2 = -e^{2\Phi}dt^2 + \frac{dr^2}{1 - 2m/r} + r^2 d\theta^2 + r^2 \sin^2\theta (d\phi - \omega dt)^2 \ , \tag{12}$$

where Φ, m and ω are functions of radius. Outside the star, $m(r) = M$ = constant is the star's mass, $e^{2\Phi} = 1 - 2M/r$ and $\omega = 2J/r^3$ where J is the star's spin angular momentum. The fluid inside the star rotates with angular velocity $\Omega = u^\phi/u^t = d\phi/dt$.

a. An observer very far from the star watches (via light or neutrinos) as a fluid element goes around and around the star's spin axis. The observer is at rest with respect to the star. Using his or her own ideal clock, the observer measures the time T that it takes for the fluid element to make one round trip and return to its starting point. Show that $T = 2\pi/\Omega$, where Ω is the fluid element's angular velocity. [Hint: Consider the trajectories in spacetime of two pulses of light or neutrinos, one emitted at the beginning of the round trip and the other at the end. How do those trajectories differ from each other?]

b. Two photons (pulses of light) travel around the star on the same circle, a path of constant r, θ; they are held on these paths by circular mirrors, along which they skim, at the vacuum speed of light. They are emitted simultaneously by an observer, one clockwise and the other counterclockwise; and the observer also moves along the same circular path (same r, θ) with an angular velocity $d\phi/dt \equiv \sigma$. Show that, in order that the simultaneously emitted photons return to the obseerver (after encircling the star) simultaneously, the observer's angular velocity must be $\sigma = \omega$.

Problems related to Spacetime Symmetries, Killing Vector Fields, and Associated Conservation Laws

6. Killing Vector Field [5 points]

Let $\vec{\xi}$ be an arbitrary vector field in an arbitrary, curved spacetime.

a. Show that it is possible to construct a coordinate system in which $\vec{\xi}$ is one of the coordinate basis vectors, $\vec{\xi} = \partial\mathcal{P}/\partial x^K$.

b. Show that in this coordinate system

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = g_{\alpha\beta,K} \ . \tag{1}$$

Thus, in this coordinate system, $\vec{\xi}$ satisfies Killing's equation if and only if the metric components are independent of the coordinate x^K, i.e., if and only if $\vec{\xi}$ is a "Killing vector field".

7. Commutator of two Killing Vectors is a Killing Vector [5 points]

Show that if $\vec{\xi}$ and $\vec{\zeta}$ are both Killing vector fields, then their commutator $[\vec{\xi}, \vec{\zeta}]$ is also a Killing vector field. [It is worth noting that, because Killing's equation is linear, $a\vec{\xi} + b\vec{\xi}$ is also a Killing vector field providing a and b are constants.]

8. Killing Vector Fields of Flat Spacetime [10 points]

Consider flat spacetime as described in a Lorentz coordinate system (t, x, y, z).

a. Write down the four Killing vector fields that generate translations along the four coordinate axes.

b. Write down the Killing vector fields that generate rotations around the x axis and the y axis. Show that their commutator is the Killing vector field that generates rotations around the z axis. Explain the relationship of this result to the commutator of the orbital angular momentum operators in quantum mechanics.

c. Flat spacetime also is invariant under boosts along the three spatial coordinate axes. Write down the Killing vector field that generates boosts along the x

direction. Show that in the proper reference frame of an observer who is uniformly accelerated in the x-direction [MTW chapter 6; metric given by Eq. (6.18)], this Killing field is the generator of time translations.

9. **Injection Energy for a Relativistic, Non-Spinning Star** [15 points]
δA baryons are injected into a relativistic star at a radius r, where its density of total mass-energy is ρ, its pressure is P and its number density of baryons is n. Before injection, these baryons are put into the same thermodynamic state as the stellar material at r, i.e. they are given the same ρ, P and n as the material there. Show that the star's mass M, as measured by an observer far away (via Kepler's laws applied to a distant planet) increases by an amount

$$\delta M = \left(\frac{\rho + P}{n}\right) e^{\Phi} \delta A = \left(\frac{\partial \rho}{\partial n}\right)_s e^{\Phi} \delta A \ . \tag{13}$$

Here the derivative is taken, thermodynamically, at fixed entropy per baryon s, and Φ is the metric coefficient [cf. Eq. (12) above] evaluated at the radius where the baryons are injected.

Note: This is a beautiful but perhaps difficult exercise in analyzing physical processes and measurements in general relativity.

Hint: Consider a thought experiment in which an observer $\mathcal{O}$ far from the star prepares the δA baryons in the desired thermodynamic state and then bundles them up in a small package and drops them into the star, through a thin pipe so they do not run into stellar fluid. A second observer inside the star catches the package, extracting its kinetic energy of fall [kinetic energy as measured by her]. She uses some of the extracted kinetic energy to open open up the required volume $\delta V = \delta A/n$ inside the star at radius r [doing work as seen by her $\delta W = P\delta V$], and she then places the baryons into that opened up volume. She then throws the empty package and any excess energy back up to the distant observer $\mathcal{O}$. The kinetic energy of throw, as measured by her, must come from the package and the extracted kinetic energy. Observer $\mathcal{O}$ catches the package and applies the law of mass conservation to compute the star's mass increase: δM =(total mass-energy that $\mathcal{O}$ dropped into the star) − (total mass-energy that $\mathcal{O}$ caught, flying up from the star).

Problems related to proper reference frame of an accelerated observer

10. Inertial and Coriolis Forces [10 points]
MTW Exercise 13.14

11. Foucault Pendulum on the Spinning Earth [12 points]
The spacetime metric outside the spinning Earth (or any other rigidly rotating, nearly spherical body) has the following approximate form:

$$ds^2 = -\alpha^2 dt^2 + \frac{dr^2}{\alpha^2} + r^2 d\theta^2 + \varpi^2 (d\phi - \omega dt)^2 \ , \tag{7}$$

where

$$\alpha = \sqrt{1 - 2M/r} \ , \quad \varpi = r \sin\theta \ , \quad \omega = 2S/r^3 \ , \tag{8}$$

and M and S are the Earth's mass and spin angular momentum. The angular momentum is caused by the Earth's rotation, which has angular velocity $\Omega = d\phi/dt$ about the axis $\theta = 0$.

a. The quantity ω is the angular velocity of "dragging of inertial frames" caused by the Earth's spin. Evaluate ω numerically and compare it with Ω; show thereby that $\omega \ll \Omega$. This justifies your treating ω as a tiny, linear correction to Ω, i.e. it justifies dropping terms that are quadratic in ω throughout this problem.

Vladimir Braginsky, Alexander Polnarev and I long ago proposed using a Foucault pendulum at the earth's South pole to measure the dragging of inertial frames [*Physical Review Letters*, **53**, 863-866 (1984)]. In this experiment a telescope is used to establish a local proper reference frame that is tied to the distant stars, and the pendulum's motion is measured relative to that proper reference frame. The experiment is to be done at the south pole rather than the north because of the high altitude, the clearness and dryness of the air (making for good astronomical "seeing"), and the presence of a large scientific base there. For conceptual and computational simplicity, however, we shall analyze the experiment as though it were at the north pole.

b. Place the origin of the telescope-transported proper reference frame precisely on the Earth's north pole, at $(r = R, \theta = 0)$. The spatial axes of the proper reference frame point toward the "fixed" distant stars. Explain why this allows us to choose for the frame's spacetime axes $\vec{e}_{\hat{0}} = \alpha^{-1}\partial/\partial t$, $\vec{e}_{\hat{x}} = R^{-1}(\partial/\partial\theta)_{\phi=0}$, $\vec{e}_{\hat{y}} = R^{-1}(\partial/\partial\theta)_{\phi=\pi/2}$, $\vec{e}_{\hat{z}} = \alpha\partial/\partial r$.

c. Compute this frame's 4-acceleration $\vec{a} = \nabla_{\vec{e}_{\hat{0}}}\vec{e}_{\hat{0}}$; and by computing $\nabla_{\vec{e}_{\hat{0}}}\vec{e}_{\hat{j}}$, determine its angular velocity $\vec{\omega}$ with respect to local "inertial-guidance" gyroscopes. Express both $\vec{a}$ and $\vec{\omega}$ in terms of the proper reference frame's basis.

d. The Foucault pendulum's plane of swing will remain fixed relative to local inertial-guidance gyroscopes. Use this fact and the result of part c to deduce the angular velocity of rotation of the pendulum's plane with respect to the distant stars, as measured in the local proper reference frame. [Your answer should be

$$\frac{\omega}{\alpha} = \frac{1}{\sqrt{1 - 2M/R}}\frac{2S}{R^3}\ . \tag{9}$$

The $1/\alpha$ is the gravitational redshift factor which relates the ticking rates of clocks in the local proper reference frame to the ticking rates far from Earth. If one measures time via the distant clocks, this angular velocity becomes $d\phi/dt = \omega$.]

e. Evaluate this result numerically, in units of milliarcseconds per year.

Discussion: This experiment is feasible with current technology, but very difficult; see the paper by Braginsky, Polnarev and me for a detailed discussion of its feasibility. Nobody has ever attempted it.