Ph 236a: General Relativity 31 October 2007

WEEK 6: Schwarzschild Black Holes, and Wormholes

Recommended Reading:

1. Blandford and Thorne, Secs. 25.2, 25.3.1, 25.4, 25.6 of version 0625.1.pdf, available on the website http://www.pma.caltech.edu/Courses/ph136/yr2006/text.html . This discusses Schwarzschild black holes and gravitational implosion to form a black hole, but not wormholes.
2. MTW Chapter 31 (Schwarzschild Geometry), Chapter 32 (Gravitational Collapse), and Section 34.2 ("Infinity in Asymptotically Flat Spacetime").
   Note: Section 32.4 provides an exact solution for the interior of an imploding star. It relies on something we have not yet studied: a Friedmann cosmological model with hyperspherical geometry for its homogeneous space slices. I do not expect you to comprehend fully the details of this section, though if you choose to study Boxes 27.1 and 27.2 of MTW you can then understand fully.]
3. S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Space-Time*, last part of Sec. 5.5 on the Reissner-Nordström solution. [A copy will be on our web site by Thursday evening.] Briefer and less technical treatments of this material are contained in MTW Exercises 31.8, 32.1 and 34.3 and in Carroll (Ref. 6 below). A more complete treatment is in Ref. 8 below.
   Note: In the copy on our course website, I have inserted abolute value signs in various formulae on page 157 to correct typos.

Possible Supplementary Reading:

4. Hartle *Gravity*: Chapter 12. This is an especially good treatment of Schwarzschild black holes and wormholes — significantly more lucid than MTW, I think.
5. Schutz (*A First Course in General Relativity*): pp. 288–297 on Schwarzschild solution and black holes.
6. Carroll *Spacetime and Geometry*: Secs. 5.1, 5.2, 5.3, 5.6, 5.7, 5.8 on Schwarzschild wormholes and black holes; and Sec. 6.5 on the Reissner-Nordstrom Solution.
7. Wald (*General Relativity*): Secs. 6.1 and 6.4.
8. J. C. Graves and D. R. Brill, "Oscillatory Character of Reissner-Nordstrom Metric for an Ideal Charged Wormhole", *Physical Review*, **120**, 1507–1513 (1960). This was the first paper to deduce the causal structure of Reissner-Nordström. It carries the analysis up to the analog of Kruskal-Szekeres coordinates, but does not do the final step of bringing infinity in to a finite location via a conformal transformation. Penrose had not yet invented that idea when this paper was written. Nevertheless, this paper might be more accessible and understandable than the much terser discussion in Hawking and Ellis (Ref. 1 above). Historically, this work was the undergraduate, Senior thesis of John Graves at Princeton in 1959; Dieter Brill, then a grad student just finishing his PhD, was Graves' advisor.

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Problems

Note: Each problem is worth 10 points unless otherwise indicated. As usual, the maximum number of points that will be given for this set is 50.

1. **Interpretation of a Metric and Coordinate System**
   A spacetime has the following metric

$$ds^2 = -\frac{dt^2}{t_o/t - 1} + (1 - t_o/t)dz^2 + t^2(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{1}$$

   Here t_o is a constant, and t increases upward from 0.
   a. What are the coordinates being used?
   b. What symmetries does this spacetime have, and how are the coordinates related to those symmetries?
   c. Without solving the geodesic equation, explain on the basis of symmetries why the curves $\{z, \theta, \phi\}$ = constant are geodesics, and why these geodesics are the world lines of freely falling observers.
   d. Plot, as a function of their proper time, the proper distances between two freely falling observers with the same $\{\theta, \phi\}$ but slightly different z. Do the same for observers with the same z but slightly different $\{\theta, \phi\}$. Discuss the physical meaning of these plots.
   e. At what values of the coordinates does the metric (1) become singular?
   f. What is the relationship of this spacetime to the Schwarzschild spacetime?
   g. Based on that relation, what is the physical nature of each of the singularities you identified in part e?

2. **The Bertotti-Robinson Solution of the Einstein Field Equation**
   Blandford and Thorne Exercise 25.2.

3. **Gore at the Singularity**
   Blandford and Thorne Exercise 25.8.

4. **Slices of Simultaneity in Schwarzschild Spacetime**
   Blandford and Thorne, Exercise 25.12.

5. **Nonradial Light Cones** [5 Points]
   MTW Exercise 31.2.

6. **Eddington-Finkelstein and Kruskal-Szekeres Compared**
   MTW Exercise 31.5.

7. **Rindler and Kruskal-Szekeres Compared**
   Divide Minkowski spacetime up into four regions bounded by leftward and rightward traveling light rays, like the four regions of Kruskal-Szekeres. In region I introduce Rindler coordinates, defined as follows: If the Lorentz coordinates are denoted

(T, X, Y, Z) and the Rindler coordinates are (t, x, Y, Z), then the transformation to Rindler in region I is

$$X = x \cosh(gt) \ , \quad T = x \sinh(gt) \ , \tag{1}$$

where g is a constant acceleration. Notice that these Rindler coordinates are nothing but the proper reference frame of a uniformly accelerated observer [Eq. (6.17) of MTW], with the spatial origin moved: $t = \xi^{0'}$, $x = \xi^{1'} + g^{-1}$. Notice also the similarity to Eq. (31.17a) for Kruskal-Szekeres coordinates.

a. Show that the spacetime metric in Rindler coordinates is

$$ds^2 = -g^2 x^2 dt^2 + dx^2 + dY^2 + dZ^2 \ . \tag{2}$$

b. Construct Rindler coordinates for regions II, III, and IV, by analogy with Eqs. (31.17b,c,d).

c. Derive the Minkowski spacetime metric in Rindler coordinates for all four regions from these transformations.

d. Draw coordinate diagrams using the Lorentz coordinates and using the Rindler coordinates, and explore the relationships between them in the same manner as MTW explores the relationship between Kruskal-Szekeres and Schwarzschild.

8. Rindler Approximation to the Schwarzschild metric near the horizon of a black hole

[Note: Again, I did this in my Monday lecture, but you may find it useful to do the details yourself.]

Build the proper reference frame of a static observer who is at rest at location (r_o, θ_o, ϕ_o) very slightly above the horizon of a Schwarzschild black hole. Introduce slightly different normalizations for the coordinates than usual. In particular, use as the time coordinate Schwarzschild time t rather than the observer's proper time τ, and use for the vertical spatial coordinate the proper distance x measured from the hole's horizon rather than from the observer's location. More specifically, use as coordinates t, $x = \int_{2M}^{r} dr/\sqrt{1 - 2M/r} \simeq 4M\sqrt{1 - 2M/r}$, $Y = r_o(\theta - \theta_o) \simeq 2M(\theta - \theta_o)$, $Z = r_o \sin\theta_o(\phi - \phi_o) \simeq 2M \sin\theta_o(\phi - \phi_o)$. Show that in these modified "proper reference frame" coordinates the Schwarzschild metric in the observer's vicinity takes on the Rindler form (2), where the "acceleration" g [which is measured in units of Schwarzschild time t rather than the observer's proper time τ] is given by $g = 1/4M$. This $g = 1/4M$ can be thought of as the hole's "surface gravity" as measured in Schwarzschild time t units.

9. Redshift During Collapse

MTW Exercise 32.2(a). *Hint:* It may be helpful to proceed as follows:

(i) For simplicity consider a star whose surface falls inward on a timelike geodesic of the Schwarzschild geometry, with $\tilde{E} = 1$. (If you are ambitious you can do the derivation for an arbitrary timelike inward world line of the surface, and show that the result is independent of the details of that world line.) Compute the redshift as a function of the radius $r_{\rm em}$ at which the photon is emitted; your result should be $\lambda_{\rm rec}/\lambda_{\rm em} \propto 1/(1-2M/r_{\rm em})$. [There is no square root! This includes both a gravitational redshift and a Doppler shift, each with a square root].

(ii) Show that the Schwarzschild time of emission of the photon is $t_{\rm em} = -2M\ln(1-2M/r_{\rm em}) + \text{constant}$.

(iii) Show that the Schwarzschild time of reception of the photon by the distant observer is $t_{\rm rec} = -4M\ln(1-2M/r_{\rm em}) + \text{constant}$. Then combine this with the result of (i) to get Eq. (32.8a) of MTW.

10. Collapse of an Electrically Charged Star

Attached to the back of this problem set is an extract from my popular book *Black Holes and Time Warps: Einstein's Outrageous Legacy* (pp. 456–458). This extract describes the evolution of the spacetime inside and around a spherically symmetric, electrically charged star that collapses, forms a black hole, then bounces and reexplodes into another universe.

a. Draw, on a Penrose diagram for the Reissner-Nördstrom spacetime, the world line of the collapsing, bouncing, and reexploding star. The region outside this world line is correctly described by the Reissner-Nördstrom metric; the region inside must be described by a different, interior solution to the Einstein field equations.

b. On your Penrose diagram, draw the spacelike hypersurfaces corresponding to each of the embedding diagrams of Figure 13.4 of *Black Holes and Time Warps*.

c. Discuss, using the Penrose diagram, the various possible subsequent evolutions of the collapsing and bouncing star.

d. Suppose that all the regions I of the Penrose diagram for this collapsing and bouncing star are the same external universe, but with their black-hole mouths at different spatial locations. Show that the stellar collapse and bounce can be used as a time machine for backward time travel — i.e., the spacetime has closed timelike curves (CTC's).