WEEK 8: THE LINEAR APPROXIMATION TO GENERAL RELATIVITY, NEWTONIAN GRAVITY, AND GRAVITATIONAL WAVE PROPAGATION

Recommended Reading:

1. For a brief summary of Linearized Theory: Section 24.9.2 of version 0624.1.K.pdf of Chapter 24 of Blandford and Thorne; available at
   http://www.pma.caltech.edu/Courses/ph136/yr2004/ .
2. Linearized Theory: MTW Chapter 18.
3. Post-Linear Expansions: After promising in class today that I would give you reading on the Post-Linear Expansion of General Relativity, I now realize that I've not yet taught you enough. We first need to formulate general relativity as a nonlinear field theory in flat spacetime. I will do that either next week or the following (last week of the term). In the meantime, students who already have enough background will find the promised material in Ref. 11 below.
4. The Newtonian Approximation and conceptual foundations for the Post-Newtonian Expansion of General Relativity: MTW Sections 39.4–39.7. These sections use a Newtonian potential $U \equiv -\Phi$ with the opposite sign from the one, Φ, that I am using in this course.
5. Gravitational Wave Propagation in Linearized Theory: MTW Secs. 35.1–35.6.

Optional Supplementary Reading:

6. Hartle, Sections 21.5 on Linearized Theory and 16.1–16.3 on Gravitational Wave Propagation
7. Schutz, Sections 8.3, 8.4 and 9.1. This is a brief, elementary introduction to Linearized Theory, the Newtonian Approximation, and Gravitational Wave Propagation.
8. Carroll, Sections 7.1–7.4 on Linearized Theory, Newtonian Approximation and Gravitational Wave Propagation. The treatment of Linearized Theory includes other gauges besides Lorenz.
9. Wald, Section 4.4. A sophisticated treatment of Linearized Theory, Newtonian Approximation and Gravitational Wave Propagation.
10. MTW Exercise 7.3 and Box 7.1, and Section 5 of Box 17.2 (pages 424, 425). This material describes Linearized Theory as the canonical spin-two field theory in flat spacetime. For a historical overview of this topic and its role in the derivation of the Einstein field equations, see the Foreward, by John Preskill and Kip Thorne, to the *Feynman Lectures on Gravitation* by R. P. Feynman, F. B. Morinigo, and W. G. Wagner (Addison-Wesley, 1995).
11. Sections I, II, and III of K. S. Thorne and S. J. Kovács, "The Generation of Gravitational Waves. I. Weak-Field Sources", *Astrophysical Journal*, **200**, 245–262 (1975). This reference develops the foundations for the "Post-Linear" (also sometimes called "Post-Minkowski") nonlinear iteration scheme for the Einstein field equations $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ and the equations of motion $T^{\alpha\beta}{}_{;\beta} = 0$. This scheme shows nonlinearities entering (i) the equations of motion and then (ii) the field equations at alternate orders.

Problems Each problem is worth 10 points unless otherwise indicated

1. **Gauge Invariance of the Riemann Curvature Tensor** [5 Points]
   MTW Exercise 18.1.

2. **Justification of Lorenz Gauge** [5 Points]
   MTW Exercise 18.2.

3. **External Field of a Static, Spherical Body in Lorenz Gauge**
   a. MTW Exercise 18.3, Part (a)
   b. MTW Exercise 18.3, Part (b)
   c. Show that the linearized spacetime metric obtained part (b) is identical to the linearized Schwarzschild metric written in isotropic coordinates (Problem 7 of Week 6). Thus, the Lorenz gauge condition automatically puts the linearized Schwarzschild solution into isotropic coordinates, rather than Schwarzschild coordinates.

4. **Stress-Energy Tensor, Equation of Motion, and Gravitational Field for Point Masses, Including Photons** [20 Points]
   By analogy with Eq. (2) of Problem 7 of Week 3, we can write the stress-energy tensor for a point particle with nonzero rest mass m, moving through curved spacetime along a world line $\mathcal{P}'(\tau)$, as

$$\mathbf{T}(\mathcal{P}) = m \int \vec{u}(\tau) \otimes \vec{u}(\tau) \delta_4[\mathcal{P}, \mathcal{P}'(\tau)] d\tau \; . \tag{14}$$

   Here the notation (equally valid in curved spacetime or flat) is the same as in Problem 7 of Week 3. Equation (14), in fact, is the solution to Problem 8 of Week 3 in flat spacetime, and it remains correct in curved spacetime since there is no way that curvature coupling could enter into this relation.

   a. Show that the stress-energy tensor (14) can be rewritten in terms of the particle's 4-momentum $\vec{p} \equiv d\mathcal{P}'/d\zeta$ (with ζ an affine parameter) as

$$\mathbf{T}(\mathcal{P}) = \int \vec{p}(\zeta) \otimes \vec{p}(\zeta) \delta_4[\mathcal{P}, \mathcal{P}'(\zeta)] d\zeta \; . \tag{15}$$

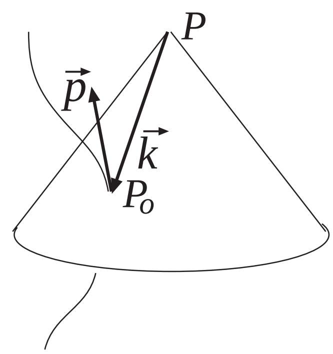
   This expression is valid for a particle with zero rest mass as well as finite rest mass (as one can infer by taking the limit as $\vec{p}$ becomes null with m going to zero.

   b. Show that the law of energy-momentum conservation in the form $T^{\alpha\beta}{}_{;\beta} = 0$ implies that the particle moves along a timelike geodesic of curved spacetime. This is a result that I claimed in my Wednesday lecture: that $\vec{\nabla} \cdot \mathbf{T} = 0$ implies the geodesic equation of motion for a point particle.

   c. Show that in Lorenz gauge the trace-reversed metric perturbation produced by this particle has the following form:

$$\bar{h}^{\alpha\beta}(\mathcal{P}) = \frac{4 p^\alpha p^\beta}{\vec{k} \cdot \vec{p}} \; . \tag{16}$$

Here $\mathcal{P}$ is the field event, $\vec{p}$ and p^α are the particle's 4-momentum evaluated at the (source) event $\mathcal{P}_o$ where the particle passes through the past light cone of $\mathcal{P}$, and $\vec{k}$ is the past-directed null vector from $\mathcal{P}$ to $\mathcal{P}_0$; see the figure:



Equation (16) is the gravitational analog of the Lienard-Weichert potential for the electromagnetic field of a point charge, as worked out in Problem 13 of Week 2.

[Hint: Begin by assuming the particle has finite rest mass, and derive $\bar{h}^{\alpha\beta}$ in the momentary rest frame of the particle. Then show that (16) is a frame-independent version of that result. And finally extend (16) to a particle with zero rest mass by taking the appropriate limit.]

d. A test particle begins at rest at the origin of a Cartesian coordinate system. A massive particle (mass m) flies past the test particle with speed v, moving in the x direction with impact parameter b, so its path is given by $x = vt$, $y = b$, $z = 0$. Evaluate $\bar{h}_{\alpha\beta}$ in the vicinity of the test particle as an explicit function of the coordinates.

e. The test particle is given a gravitational "kick" by the passing massive particle. Show that it acquires a net velocity in the y direction given by

$$\frac{dy}{dt} = \frac{2Gm}{\gamma v b}(1 + 2\gamma^2 v^2) \ , \quad \text{where } \gamma \equiv 1/\sqrt{1 - v^2} \ . \tag{17}$$

Here G is Newton's gravitational constant (normally set equal to unity). Note that, in the small v limit, the kick is as predicted by Newtonian theory, $dy/dt = 2Gm/vb$.

f. By taking the limit $m\gamma \to E$, $m \to 0$, infer that if the passing particle is a photon of energy E, the gravitational kick received by the test particle is

$$dy/dt = 4GE/b \ .$$

g. For a discussion of a possible experiment to test general relativity based on the kick (17), using as the massive, high-speed particle a bunch of protons in a storage ring, see pages 2065–2066 of Braginsky, Caves & Thorne, *Phys. Rev. D*, 2047 (1977).

5. Newtownian Approximation to General Relativity

MTW Exercise 39.3.

6. Electromagnetic Analogs of $h^{\rm TT}_{jk}$, h_+ and $h_\times$ [20 Points]
The gravitational-wave analysis given in my Wednesday lecture and in Ref. 5 above and in the exercises that follow is closely analogous to the following treatment of electromagnetic waves.

Consider a plane electromagnetic wave propagating in the z direction through a Lorentz frame of flat spacetime. The wave has an antisymmetric electromagnetic field tensor $F_{\mu\nu}(t-z)$ whose components are related to those of the electric and magnetic field by $F_{j0} = -F_{0j} = E_j$ and $(F_{23}, F_{31}, F_{12}) = (B_1, B_2, B_3)$.

a. Use Maxwell's equations to verify that **E** and **B** are transverse (have vanishing z components), and that all components of $F_{\mu\nu}$ can be expressed in terms of F_{j0}; in other words, the magnetic field can be expressed in terms of the electric field. This is analogous to the transversality of the tidal forces in a gravitational wave, and to the fact that all components of the Riemann tensor for a gravitational wave are expressable in terms of R_{j0k0}.

b. Define $A^{\rm T}_j$ by $E_j \equiv -A^{\rm T}_{j,t}$. Here and throughout we use the notation that subscripts $0, 1, 2, 3$ are equivalent to subscripts t, x, y, z. This $A^{\rm T}_j$ is the analog of $h^{\rm TT}_{jk}$ for a gravitational wave. Since the electromagnetic wave is transverse, the only nonzero components of $A^{\rm T}_j$ are $A^{\rm T}_x$ (the analog of h_+) and $A^{\rm T}_y$ (the analog of $h_\times$).

c. Now introduce the 4-vector potential A_μ (not to be confused with $A^{\rm T}_j$), from which the electromagnetic field tensor can be constructed via $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$. Show that in Lorenz gauge, where ${A_\mu}^{,\mu} = 0$, Maxwell's equations reduce to the wave equation for A_μ, and thence (since we are considering a plane wave propagating in the z direction), A_μ is a function only of $t-z$. This is the analog of the trace reversed metric peturbation for a plane gravitational wave being a function of $t-z$ in gravitational Lorenz gauge (discussed in my lectures).

d. Find a specific gauge-change generator $\Psi(t-z)$ that brings A_μ into a special Lorenz gauge in which $A_0 = A_z = 0$ so the that A_μ is *transverse*. Show that in this special Lorenz gauge, the spatial components of the vector potential are $A_j = {A_j}^{\rm T}$. We call this Transverse gauge or T gauge. It is the electromagnetic analog of TT gauge for a gravitational wave.

e. Show that the T-gauge fields ${A_x}^{\rm T}$ and ${A_y}^{\rm T}$ can be obtained from the vector potential in any gauge where $A_\mu = A_\mu(t-z)$ by simple projection — i.e., by throwing way the temporal and longitudinal components of A_μ and setting $A^{\rm T}_x = A_x$ and $A^{\rm T}_y = A_y$. This is the analog of computing the components of $h^{\rm TT}_{jk}$ by projection, in any gauge where the metric perturbation is a function of $t-z$.

f. The fields ${A_x}^{\rm T}$ and ${A_y}^{\rm T}$ depend on one's choice of reference frame. Show that when one rotates the frame's basis vectors in the transverse plane through an angle ψ, then ${A_x}^{\rm T}$ and ${A_y}^{\rm T}$ change by

$$({A_x}^{\rm T} + i{A_y}^{\rm T})_{\rm new} = ({A_x}^{\rm T} + i{A_y}^{\rm T})_{\rm old} e^{-i\psi} \ . \tag{1}$$

This equation tells us that $A^{\rm T}_j$ has spin weight one, and the photon which carries these waves must have spin one in units of Planck's constant.

g. Show that, when one performs a boost along the z axis to a new reference frame moving at speed β with respect to the old one, the fields $A_x^{\rm T}$ and $A_y^{\rm T}$ (which are defined in terms of the electric fields measured in the two frames), are unchanged at a fixed location in spacetime; i.e. they behave like scalars; i.e. they have boost weight zero. [Hint: use the result of part e.]

h. Show that the electric field E'_j measured in the boosted frame and that E_j in the original, unboosted frame are related by

$$E'_x = \sqrt{\frac{1-\beta}{1+\beta}}E_x \ , \quad E'_y = \sqrt{\frac{1-\beta}{1+\beta}}E_y \ . \tag{2}$$

The factor $\sqrt{\frac{1-\beta}{1+\beta}}$ is the Doppler shift in going from one frame to the other. Show, further, that the energy density and energy flux in the waves, $T^{00} = T^{0z} = (1/8\pi)(\mathbf{E}^2 + \mathbf{B}^2)$ behave under a boost as

$$T'^{00} = T'^{0z} = \left(\frac{1-\beta}{1+\beta}\right) T^{00} = \left(\frac{1-\beta}{1+\beta}\right) T^{0z} \ . \tag{3}$$

These equations say, in words, that E_j has boost weight one, and the energy density and energy flux have boost weight two.

7. Boost Weight and Spin Weight of h_+ and $h_\times$

a. Show that, when one rotates the spatial axes of the transverse (x, y) plane through an angle ψ, the gravitational-wave fields defined with respect to the rotated ("primed") axes are related to those defined with respect to the old ("unprimed") axes by

$$h'_+ + ih'_\times = (h_+ + ih_\times)e^{-i2\psi} \ , \tag{5}$$

i.e., $h_{jk}^{\rm TT}$ has spin weight two, so the graviton must have spin two in units of Planck's constant.

b. Show that h_+ and $h_\times$ have spin-weight zero.

8. Riemann Tensor in Terms of h_+ and $h_\times$

For a plane gravitational wave propagating in the z-direction through flat spacetime, express all the nonzero components of the Riemann tensor explicitly in terms of $h_+(t-z)$ and $h_\times(t-z)$.

9. Transformation to TT Gauge

In this exercise you will derive, in the time domain, results that MTW derive in the frequency domain—i.e. that MTW derive for monochromatic waves which can be superposed to get your time-domain results.

a. Consider any trace-reversed metric perturbation that is in Lorenz gauge. Show that a further gauge change whose generators satisfy the wave equation $\xi_{\alpha,\mu}{}^{\mu} = 0$ leaves $\bar{h}_{\alpha\beta}$ still in Lorenz gauge. Show that such a gauge change, in general, involves four free functions of *three* of the spacetime coordinates, by contrast with general gauge transformations which entail four free functions of all *four* spacetime coordinates.

b. Consider a plane gravitational wave in Lorenz gauge, propagating in the z direction, so $h_{\alpha\beta}$ is a function only of $t - z$. Exhibit gauge-change generators ξ_α that satisfy the wave equation and that remove four of the six independent functions from $\bar{h}_{\alpha\beta}$, bringing it into TT gauge, so the only nonzero components of $h_{\alpha\beta}$ are $h_{xx} = -h_{yy} = h_+(t - z)$m $h_{xy} = h_{yx} = h_\times(t - z)$.

c. Show by an explicit calculation that the gauge change of part (b) can be achieved by throwing away all pieces of $h_{\alpha\beta}$ except the transverse ones (those that lie in the x-y plane) and by then removing the trace — i.e. by "transverse-traceless projection".

10. Generation of a Gravitational Wave by an Electromagnetic Wave Beating Against a DC Magnetic Field [15 Points]

a. An electromagnetic wave with magnetic field amplitude B_o and frequency ω, propagating in the z direction, has a stress-energy tensor

$$T^{00} = T^{0z} = T^{z0} = T^{zz} = \frac{B_o^2}{4\pi} \cos^2 \omega(t - z) \ . \qquad (AA)$$

Show that this propagating electromagnetic wave does not produce any gravitational waves whatsoever. [Hint: In Lorenz gauge this $T^{\mu\nu}$ produces a nonzero $\bar{h}^{\mu\nu}$, but that trace-reversed metric perturbation is pure gauge; it has vanishing Riemann tensor, as one can infer via TT projection.]

b. The same electromagnetic wave (with $\mathbf{B_o}$ pointing in the transverse x direction) propagates through a region of space with a DC magnetic field $B_{\rm DC}$ that also points in the transverse x direction. (For example, $B_{\rm DC}$ might be an interstellar magnetic field.) Assume that $B_o \ll B_{\rm DC}$, and that the DC magnetized region is confined to $z > 0$. Show that the beating of the wave's B_o against $B_{\rm DC}$ gives rise to a stress-energy tensor that generates gravitational waves of the form

$$h_+ = \frac{2B_{\rm DC}B_o}{\omega} z \sin \omega(t - z) \ , \quad h_\times = 0 \quad \text{for } z > 0 \ . \qquad BB$$

Explain why this represents a resonant transfer of energy from the electromagnetic wave to the gravitational wave.