WEEK 9: THE GENERATION OF GRAVITATIONAL WAVES,

Recommended Reading:

1. Section 26.4 of version 0624.1.K.pdf of Chapter 26 of Blandford and Thorne; available at http://www.pma.caltech.edu/Courses/ph136/yr2004/ . This includes a small portion about the energy carried by gravitational waves [the paragraph containing Eq. (26.113)] that relies on material I will cover next week.
2. Chapter 36 of MTW, on *Generation of Gravitational Waves.*
   Notes:
   a. The discussion of specific sources of gravitational waves in Sec. 36.5 is very out of date, so don't pay much attention to it. You will learn a lot about specific sources in the second term of this course.
   b. Ignore the brief discussion of the stress-energy tensor for gravitational waves on page 992. We will study that next week.
   c. Sections 36.9 and 36.10 give a derivation of the quadrupole-moment wave generation formula using post-linear theory (though the phrase post-linear had not yet been coined). This derivation is valid for nearly Newtonian self gravitating systems such as a binary. If you set $t^{\mu\nu} = 0$, then this becomes the linearized-theory derivation given in most textbooks, e.g. those written by Carroll, by Hartle, by Schutz and by Wald, which is not valid for systems with significant self gravity such as binaries.
   d. The derivation of Burke's radiation reaction potential in Section 36.11 is flawed: it omits some nonlinear terms that are of the same magnitude as the linear ones which are kept, and that cancel each other in the end. The flaw is discussed in Walker and Will, *Astrophysical Journal*, **242**, L129 (1980). (This is one of two serious errors that I am aware of in MTW.) The first correct derivation of $\Phi^{(\text{react})}$, so far as I know, was in Section III of Thorne, *Astrophysical Journal*, **158**, 997 (1969) using "Regge-Wheeler gauge" instead of Lorenz gauge; and it includes the contributions of all mass multipoles, not just the mass quadrupole. I don't expect you to read these references!
3. K.S. Thorne, "Multipole Expansions of Gravitational Radiation", *Reviews of Modern Physics*, **52**, 299 (1980) — the following sections:
   a. Sec. I.C: Notation.
   b. Secs. II.A, II.B, II.C: STF tensors, integrations over a sphere, and scalar spherical harmonics.
   c. Last page (page 315) of Sec. II.G: Solutions of scalar, vector, and tensor wave equations in terms of STF tensors.
   d. Sec. IV.A, focusing on Eq. (4.8), which is the gravitational wave field in the wave zone of an arbitrary, isolated source written in terms of multipole moments. The mass-quadrupole term is our familiar $(2/r)\ddot{\mathcal{I}}^{\text{TT}}_{jk}(t-r)$.
   e. Sec. VIII: External gravitational field of an isolated system in linearized theory. [Notes: $\gamma^1_{\alpha\beta}$ is the trace-reversed metric perturbation $\bar{h}_{\alpha\beta}$. As I discussed in class on Wednesday, the general solution (8.12), (8.13) to the vacuum, linearized

Einstein equation is valid in the weak-field region around any isolated source, independently of how strong the source's internal gravity may be and how fast its internal motions may be. If the source has slow motion, then the multipole moments that appear in this solution can be read off of the nearly Newtonian gravitational field in the source's weak-field near zone.]

Optional Supplementary Reading:

4. Hartle, Secs. 23.4–23.8. This is an elementary discussion of the quadrupole formalism for gravitational wave generation.
5. Thorne, "Gravitational Radiation: An Introductory Review", in the book *Gravitational Radiation*, edited by N. Deruelle and T. Piran (North Holland 1983): Sections 3.1, 3.2 and 3.4. These sections summarize the most important of the material that I am suggesting you read in Ref. 3, on multipole expansions of gravitational waves. It may be easier to read than Ref. 3.

****

Problems [Each problem is worth 10 points unless otherwise indicated. The maximum number of points for this problem set is 50.]

1. **Quadrupolar wave generation in linearized theory**
   Blandford and Thorne Exercise 26.7
   Note: This is the derivation of the quadrupole wave generation formula given in most textbooks. It is valid only for sources with negligible self gravity (no back-action of the gravitational field on the source).
2. **Energy Removed by Gravitational Radiation Reaction**
   Blandford and Thorne Exercise 26.9.
   [Note: I did this calculation in my Wednesday lecture. If you felt comfortable with what I told you, then skip this problem. If you want to solidify your understanding of it, do this problem.]
3. **Gravitational Waves Emitted by a Linear Oscillator**
   Blandford and Thorne Exercise 26.11.
4. **Gravitational Waves from Waving Arms**
   Blandford and Thorne Exercise 26.12
5. **Radiation Reaction in a Binary System**
   For the binary system analyzed in Sec. 26.4.3 of Blandford and Thorne, evaluate the radiation reaction potential $\Phi^{\rm react}$ and compute the force that it exerts on each star in the binary. From that force, compute the rate at which the two stars' separation a decreases, and thence the rate at which the binary's angular velocity Ω changes. Integrate your equation for $d\Omega/dt$ to obtain $\Omega(t)$. Your result should agree with Eq. (26.127) of Blandford and Thorne.

6. **First Time Derivative of Moments of Newtonian Mass Distribution** [5 Points]
The ℓ'th moment of the Newtonian mass distribution ρ is defined by

$$I_{A_\ell} \equiv I_{a_1 a_2 \ldots a_\ell} = \int \rho x^{a_1} x^{a_2} \cdots x^{a_\ell} d^3x \equiv \int \rho X_{A_\ell} d^3x \ . \tag{1}$$

Using the result of MTW Exercise 39.4, $[d/dt \int \rho f d^3x = \int \rho (df/dt) d^3x$ for any $f(\mathbf{x}, t)$, where d/dt under the integral is the time derivative moving with the fluid], which I derived and discussed in class, show that

$$\frac{dI_{a_1 a_2 \ldots a_\ell}}{dt} = \left(\ell \int \rho v^{a_1} x^{a_2} x^{a_3} \cdots x^{a_\ell} d^3x \right)^S \tag{2}$$

where the superscript S means "symmetrize on all indices", and where v^j is the velocity of the Newtonian matter.

7. **Multipole Moments of an Isolated, Newtonian System**
In electromagnetic theory one writes the electrostatic potential for an arbitrary, time-independent charge distribution as the following sum over the distribution's multipole moments $q_{\ell m}$

$$\Phi = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} q_{\ell m} \frac{Y_{\ell m}(\theta, \phi)}{r^{\ell+1}} \ ; \tag{3}$$

see, e.g., Jackson, *Classical Electrodynamics*. Since the field equation for the electrostatic potential Φ is identical to that for the Newtonian gravitational potential, $\boldsymbol{\nabla}^2 \Phi = 4\pi\rho$ (with the mass density ρ replaced by minus the charge density), the Newtonian gravitational field outside an isolated, time-independent mass distribution must also be expressible in the form (3). From the relationship between spherical harmonics $Y_{\ell m}(\theta, \phi)$ and STF tensors discussed in Ref. 3.b above, it should be evident that we can rewrite Eq. (3) in the gravitational case (with sign change) as

$$\Phi = -\sum_{\ell=0}^{\infty} \frac{(2\ell-1)!!}{\ell!} \mathcal{I}_{A_\ell} \frac{N_{A_\ell}}{r^{\ell+1}} \ , \tag{4}$$

where the constant STF tensors $\mathcal{I}_{A_\ell}$ are the multipole moments and the ℓ-dependent coefficient has been adjusted to produce a desired normalization for the moments; see Eq. (7) below.

a. Verify that expression (4) satisfies the vacuum Newtonian field equation $\boldsymbol{\nabla}^2 \Phi = 0$.
b. As a first step in deriving an expression for the multipole moment $\mathcal{I}_{A_\ell}$ as an integral over the mass distribution, show that

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \sum_{\ell=0}^{\infty} \frac{(2\ell-1)!!}{\ell!} \frac{N_{A_\ell}}{r^{\ell+1}} (X'_{A_\ell})^{STF} \ , \tag{5}$$

where the notation is that of Ref. 3 above. [Note: there are two ways to derive this expression. The first is to write $|\mathbf{x} - \mathbf{x}'| = \sqrt{(r\mathbf{n} - \mathbf{x}')^2} = \sqrt{r^2 - 2n_i x'_i + x'_j x'_j}$

and then perform a Taylor expansion about the origin. The second is to argue that (i) the left side of (4) has vanishing Laplacian, so the term of order $1/r^{\ell+1}$ on the right side must also have vanishing Laplacian; (ii) from this argue that the $1/r^{\ell+1}$ term must have the form shown in Eq. (5) but with an as yet unknown ℓ-dependent coefficient; (iii) compute the coefficient.]

c. From Eq. (5) and the general solution to the Newtonian field equation

$$\Phi = -\int \frac{\rho' d^3x'}{|\mathbf{x} - \mathbf{x}'|} , \tag{6}$$

derive the multipolar expansion (4) and show that the ℓ-pole moment $\mathcal{I}_{A_\ell}$ is the STF part of the ℓ'th moment of the mass distribution; cf. Eq. (1).

$$\mathcal{I}_{A_\ell} = (I_{A_\ell})^{STF} . \tag{7}$$

8. Current Multipoles of a Nearly Newtonian System

Consider a nearly Newtonian system, for which Linearized theory tells us the following expression for the off-diagonal time-space part of the metric:

$$g_{0j} = h_{0j} = \bar{h}_{0j} = -4\int \frac{\rho' v'_j}{|\mathbf{x} - \mathbf{x}'|} d^3x' . \tag{8}$$

Show that, if the mass distribution is time-independent so the first time derivatives of all its moments vanish, $dI_{A_\ell}/dt = 0$ for all ℓ, then

$$g_{0j} = -\sum_{\ell=1}^{\infty} \frac{4\ell(2\ell-1)!!}{(\ell+1)!} \epsilon_{jka_\ell} \mathcal{S}_{kA_{\ell-1}} \frac{N_{A_\ell}}{r^{\ell+1}} , \tag{9}$$

where $\mathcal{S}_{A_l}$ is the STF part of the $(\ell-1)$'th moment of the mass distribution

$$\mathcal{S}_{A_l} = \left(\int \epsilon_{a_1 pq} x_p \rho v_q x_{a_2} ... x_{a_\ell} d^3x \right)^{STF} . \tag{10}$$

[*Hints:* (i) Combine Eqs. (5) and (8) to obtain g_{0j} as a power series in $1/r$; call that expression (A). (ii) Combine Eqs. (9) and (10), and contract the two Levi-Civita tensors against each other to get antisymmetric products of terms under the integral. Manipulate the resulting integral, using the vanishing value of (2), to bring it into the same form as expression (A).]

9. Relativistic Multipole Moments of a Nearly Newtonian System in General Relativity

Consider a nearly Newtonian system. Its external, general relativistic gravitational field must have the general linearized form derived in Ref. 3 above. In particular, $\bar{h}_{\alpha\beta}$ (called $\gamma^1_{\alpha\beta}$ in Ref. 3) must have the form (8.12). In this exercise we shall show that the moments which appear in this relativistic gravitational field are the same as the ones we have studied in the above exercises using Newtonian and nearly Newtonian theory.

Since the system is nearly Newtonian, the timescale $\mathcal{T}$ on which its relativistic moments change is far larger than the system's size $\mathcal{L}$, which means that the body is confined to the deep interior of its near zone. Using this fact, and using Eq. (2.53b) of Ref. 3, show that near the body its relativistic gravitational field $\bar{h}_{\alpha\beta}$ [Eqs. (8.12) of Ref. 3] takes the form

$$\bar{h}_{00} \simeq 4 \sum_{\ell=0}^{\infty} \frac{(2\ell-1)!!}{\ell!} \mathcal{I}_{A_\ell} \frac{N_{A_\ell}}{r^{\ell+1}} , \tag{11}$$

$$\bar{h}_{0j} \simeq -\sum_{\ell=1}^{\infty} \frac{4\ell(2\ell-1)!!}{(\ell+1)!} \epsilon_{jka_\ell} \mathcal{S}_{kA_{\ell-1}} \frac{N_{A_\ell}}{r^{\ell+1}} . \tag{12}$$

$$\bar{h}_{ij} \simeq 0 . \tag{13}$$

By comparing with Eqs. (4) and (9), show that the system's relativistically defined moments $\mathcal{I}_{A_\ell}$ and $\mathcal{S}_{A_\ell}$ are the same as the moments that we derived in Newtonian and nearly Newtonian theory as integrals over the source, i.e. are given by Eqs. (7), (1) and (10).

10. **Reconstruction of the Full Linearized Gravitational Field from Its Gravitational Waves**. [20 Points]
[This problem is to give you practice at manipulating solutions of the wave equation using STF representations of spherical harmonics.] The most general gravitational-wave field in a source's asymptotic rest frame can be expanded in multipole moments as follows [Eq. (4.8) of Ref. 3]:

$$h^{\rm TT}_{jk} = \sum_{l=2}^{\infty} \left[\frac{4}{l!} \frac{{}^{(l)}\mathcal{I}_{jk\,L-2}(t-r) N_{L-2}}{r} + \frac{8l}{(l+1)!} \frac{\varepsilon_{pqj}\, {}^{(l)}\mathcal{S}_{kp\,L-2}(t-r) n_q N_{L-2}}{r} \right]^{\rm S\,TT} \tag{14}$$

Some comments on notation: The prefixed superscript (l) means to take l time derivatives of the multipole moment that follows. $\mathcal{I}_{jk\,L-2}$ is short-hand notation for the radiation field's STF mass l-pole moment, with its last $l-2$ indices, $a_1 a_2 ... a_{l-2}$ denoted $L-2$. N_{L-2} is short-hand notation for $n_{a_1} n_{a_2} ... n_{a_{l-2}}$; and correspondingly the last $l-2$ indices of $\mathcal{I}$ are contracted on the $l-2$ indices of N. The notation for the radiation field's current l-pole moment is similar. The superscript S means to symmetrize on the free indices (j and k); the superscript TT means to project out the transverse traceless part. [Note: In Ref. 3 and in previous problems, the notation A_{l-2} is used rather than $L-2$; it's just a matter of taste; take your choice.]

a. Construct an exact solution $\bar{b}_{jk}$ of the flat-space wave equation $\bar{b}_{jk\,,\mu\nu}\eta^{\mu\nu} = 0$ which, in the local wave zone, reduces to the above gravitational-wave field, $\bar{b}_{jk} = h_{jk}$ [Eq. (14) with the TT removed]. Your answer should be the same as Eq. (8.12a) of Ref. 3.

b. Construct the $\bar{b}_{j0}$ and $\bar{b}_{00}$, which together with your $\bar{b}_{jk}$, forms a general solution of the linearized vacuum field equations and Lorenz gauge condition. Your answer should be the same as (8.12b,c) of Ref. 3.

c. Consider a slow-motion source. In the source's weak-field, near zone, compute the leading-order part of $\bar{b}_{\alpha\beta}$. Show that, when reexpressed in Newtonian language, it is a Newtonian gravitational field written in the standard multipolar expansion (except that the monopolar $-M/r$ term is missing because it cannot be inferred from the radiation field—our starting point):

$$\Phi = \frac{1}{2} b_{00} = \sum_{l=2}^{\infty} \frac{-(2l-1)!!}{l!} \frac{\mathcal{I}_L N_L}{r^{l+1}} \tag{15}$$

[Eqs. (4) and (11) above]. This Newtonian field does not know about the current moments $\mathcal{S}_L$. They show up most prominently in the space-time part of the near-zone metric perturbation; show that it takes the following form:

$$b_{0j} = \sum_{l=1}^{\infty} \frac{-4l(2l-1)!!}{(l+1)!} \frac{\epsilon_{jka_l} \mathcal{S}_{k\,L-1} N_L}{r^{l+1}}$$

[Eq. (12) above].