Ph 236a: General Relativity 21 November 2008

WEEK 8: Kerr Black Holes
Guest lecturer: Yasushi Mino
Due date: Tuesday, December 2

Recommended Reading:

1. A quick introduction to Kerr black holes: Blandford and Thorne, Chapter 25 (version 0625.1.K.pdf): Section 25.5.
   Note that the coordinates (25.79) referred to as "a variant of a coordinate system originally introduced by Kerr" are a slightly different variant from the one used in Misner, Thorne and Wheeler, *Gravitation*. Both variants remove the coordinate singularity at the horizon; they just do it in a slightly different manner.
2. An elementary introduction to Kerr black holes, including particularly nice treatments of the Penrose Process and Blandford-Znajek Process for extracting a hole's rotational energy, and orbits in the equatorial plane of a Kerr hole: Hartle, *Gravity: An Introduction to Einstein's General Relativity*, Chapter 15.
3. The maximal analytic extension of the Kerr spacetime, with its infinity of universes: Hawking and Ellis, *The Large Scale Structure of Space-Time*, Sec. 5.6.
4. Misner, Thorne and Wheeler, *Gravitation*, Chapter 33 on Kerr black holes. In this chapter, I suggest you ignore the electric charge of the hole: set it to zero. This chapter is rather out of date (it was written in the era when Kerr black holes were just beginning to be understood), but the following portions might be useful:
   a. Box 33.2, which summarizes the mathematics of the Kerr metric in both Boyer-Lindquist coordinates and ingoing Kerr coordinates. Note that the *null* Kerr time coordinate $\tilde{V}$ used in MTW is related to that $\tilde{t}$ used in my Monday lecture by $\tilde{V} = \tilde{t} + r$; my $\tilde{t}$ is introduced in Exercise 33.5 of MTW, which is assigned below.
   b. Sec. 33.4, on symmetries and frame dragging.
   c. Sec. 33.5 on equations of motion for test particles.
   d. Sec. 33.7 on storage and removal of energy from black holes.
   e. Sec. 33.8 on reversible and irreversible transformations for black holes

Optional Supplementary Reading:

5. Hartle, Secs. 9.3 and 9.4 on geodesic orbits around a Schwarzschild black hole.
6. Schutz, Secs. 11.3 and 11.4 on Kerr black holes and Hawking radiation.
7. Wald, Secs. 12.3, 12.4, 12.5, and 14.4. This treats the Kerr metric in a more sophisticated way than the above material.
8. Sean Carroll, *Spacetime and Geometry*, Sec. 6.6 on the Kerr solution, and Sec. 6.7 on the Penrose Process.
9. On black-hole magnetospheres and the Blandford-Znajek process for extracting energy from Kerr black holes: Box 15.1 of Hartle; Chapter 8 of V.P. Frolov and I.D. Novikov, *Black Hole Physics*, second edition (Kluwer, 1998).

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Problems Note: Each problem is worth 10 points unless otherwise indicated. The maximum number of points that will be given for this problem set is 50.

1. **Blandford & Thorne's Form of Kerr Metric Compared to Form in Standard Textbooks** [5 Points]

   Show that Eqs. (25.70) of Blandford and Thorne [which is also what Yasushi wrote down during his lectures] is equivalent to the one found in most standard textbooks [Eqs. (33.2)–(33.3) of MTW with electric charge $Q = 0$].

2. **Dischronal Region of Kerr Spacetime** [Each part is worth 5 points; the full problem is worth 10 points]
   a. Consider the Kerr spacetime with $0 < a < M$. Focus on one of the regions III in the conformal diagram of Fig. 28 of Hawking and Ellis. This region is covered by a coordinate patch $-\infty < r < r_-$ of the Boyer-Lindquist coordinates (t, r, θ, ϕ). Show that at $\theta = \pi/2$ and r negative but very small in magnitude ($-M < -a \ll r < 0$), Σ^2 and ϖ^2 are negative. Show that, correspondingly, the circle of constant (t, r, θ) and $0 \leq \phi \leq 2\pi$ that passes through this point is a closed, timelike curve (CTC). Note that this CTC is near the ring singularity ($r = 0, \theta = \pi/2$). Call this CTC $\mathcal{C}_1$.
   b. Show that through any event $\mathcal{P}$ in region III there passes a CTC. Do so by sending a timelike but nearly null curve from $\mathcal{P}$ to the circular CTC $\mathcal{C}_1$, by then sending it along a timelike path that is nearly identical to $\mathcal{C}_1$ but that moves slightly backward in coordinate time t with each circuit (show that this is possible), by thereby sending it arbitrarily far back in t, and by then sending it to its starting point $\mathcal{P}$ along a timelike but nearly null curve.

3. **Infall of Particle into Kerr Black Hole**
   a. Consider a particle that falls into a Kerr black hole, beginning at rest far from the hole in the hole's equatorial plane. Derive an ordinary differential equation for the orbit of this particle, $d\phi/dr =$ (some function of r).
   b. Integrate this equation numerically for values $a/M = 0$, 0.5, 0.95, and 1, and graph your solution. Your graphs should show the particle being pulled into circular motion around the black hole (motion with angular velocity Ω_H) as it approaches the horizon.

4. **Light Cones in Ingoing Kerr Coordinates**

   Exercise 33.5 of Misner, Thorne and Wheeler.

5. **ZAMOs and Horizon Generators** [15 points]
   a. Consider a zero-angular-momentum observer (ZAMO) who hovers above a Kerr black hole moving along the world line $\phi = \omega t$, $\{r, \theta\}$ =constant. Show that this ZAMO has 4-velocity

   $$\vec{u} = -\alpha \boldsymbol{\nabla} t \,, \tag{1}$$

   i.e., the only nonzero covariant component of the ZAMO 4-velocity is $u_t = -\alpha$. Here α is the Kerr-metric lapse function. Note that this implies that the ZAMO world lines are orthogonal to the hypersurfaces of constant Boyer-Lindquist time t =constant.
   b. Show that the 4-acceleration of a ZAMO, $\vec{a} \equiv \nabla_{\vec{u}} \vec{u}$, is equal to

   $$\vec{a} = \vec{\nabla} \ln \alpha \,. \tag{2}$$

c. Show that in the limit as the ZAMO is arbitrarily close to the horizon, the magnitude of the ZAMO's 4-acceleration is

$$a = \frac{\kappa_H}{\alpha} \; , \tag{3}$$

where

$$\kappa_H = \frac{r_H - M}{2Mr_H} \; . \tag{4}$$

[Note: in Blandford and Thorne and in my lectures the notation g_H is used for κ_H.]

d. Show that in the limit as one approaches the horizon, the ZAMO 4-velocity $\vec{u}$ satisfies $\alpha\vec{u} \to \vec{\ell} \equiv \partial/\partial t + \Omega_H \partial/\partial\phi$. By deriving an analogous limit for $\alpha^2\vec{a}$ (where $\vec{a}$ is the ZAMO 4-acceleration), show that the tangent $\vec{\ell}$ to the Horizon generators satisfies $\nabla_{\vec{\ell}}\vec{\ell} = \kappa_H \vec{\ell}$.

e. The parameter $\tilde{t}$ and tangent vector $\vec{l}$ are natural ones to use for the generators since they are tied to the rate of flow of time at infinity ($\tilde{t} = t$ there) and to the symmetries of spacetime ($\vec{l}$ is the value of the Killing vector field $\partial_{\tilde{t}} + \Omega_H \partial_{\tilde{\phi}}$ at the Horizon, the location where this Killing vector field is null and all other Killing vector fields are spacelike). Another natural parametrization and tangent vector are $\zeta \equiv e^{\kappa_H \tilde{t}}$ and $\vec{k} \equiv d/d\zeta$. Show that

$$\nabla_{\vec{k}}\, \vec{k} = 0 \; ; \tag{5}$$

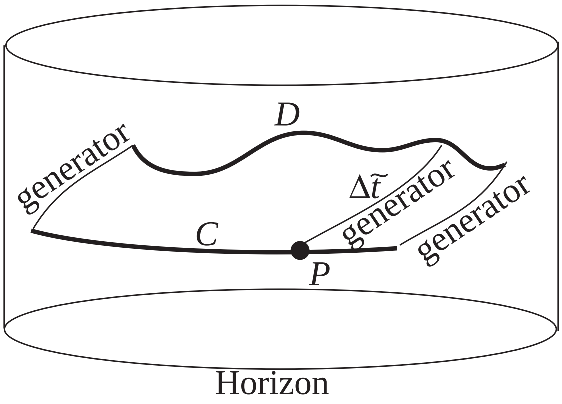
i.e., ζ is an affine parameter, and the generator is a null geodesic. Thus, $\nabla_{\vec{l}}\,\vec{l} = \kappa_H \vec{l}$ was actually the geodesic equation in disguise — disguised because the geodesic was parametrized by a non-affine parameter $\tilde{t}$.

6. Surface Area of Horizon

a. The intersection of the spacelike hypersurface $\tilde{t} =$ constant and the black-hole horizon $r = r_H$ is a 2-dimensional surface. We can think of this 2-surface as the horizon at the moment of time $\tilde{t} =$ constant. Write down the 2-metric of this 2-surface and show that its total area (the "surface area" of the horizon at time $\tilde{t}$) is $A_H = 4\pi(r_H^2 + a^2)$.

b. Through each event $\mathcal{P}$ on the horizon passes one generator, with tangent vector $\vec{l}$. Let $\vec{A}$ be an arbitrary vector at $\mathcal{P}$ that lies in the horizon. Show that $\vec{A} \cdot \vec{l} = 0$ [hint: this is true for any null surface; one way to prove it is in a suitable local Lorentz frame]. Thus, $\vec{l}$ is orthogonal to every vector that lies in the horizon at $\mathcal{P}$. This means that $\vec{l}$ is the normal to the horizon, even though it is also tangent to the horizon—a weird situation that is possible because the horizon is a null surface.

c. Let $\mathcal{C}$ be an arbitrary curve in the horizon at time $\tilde{t} =$ constant, i.e. in the 2-surface ($r = r_H, \tilde{t} =$ constant. Deform $\mathcal{C}$ into a new curve $\mathcal{D}$ by carrying each point $\mathcal{P}$ on it forward along a generator by some arbitrary amount of time $\Delta\tilde{t}(\mathcal{P})$. Show that the length of the deformed curve $\mathcal{D}$ is identically the same as the length of the original curve $\mathcal{C}$.



d. By analogy with part c., consider an arbitrary spacelike slice through the horizon, i.e. the intersection of the horizon with an arbitrary spacelike 3-surface $\mathcal{S}_3$. This intersection is a 2-surface that can be thought of as the horizon at the moment of "time" represented by $\mathcal{S}_3$. Show that the surface area of this horizon 2-surface on $\mathcal{S}_3$ is identical to that, A_H of the horizon at time $\tilde{t}$ (part a.). Thus, the horizon's surface area is independent of how one slices through it.

e. Show that the 2-metric of the 2-surface in part d. is the same as that of the 2-surface of constant $\tilde{t}$ in part a.

7. Magnetized Accretion Disk [20 Points]

This exercise will give you an opportunity to see for yourself how, in a Kerr spacetime, Maxwell's equations behave as seen by the family of ZAMOs, whose world lines are $r=$const, $\theta=$const, $d\phi/dt = \omega =$ (angular velocity of frame dragging).

Consider a rotating black hole surrounded by an accretion disk. Threading through the disk are magnetic field lines. Idealize the gas of the disk as perfectly electrically conducting, so that in the mean rest frame of the gas the electric field vanishes — which means in turn that the magnetic field lines are "frozen into the gas" (cf. any text on plasma physics or magnetohydrodynamics). The ZAMOs have, as their orthonormal basis vectors, the following, which are expressed in terms of the basis vectors of Boyer-Lindquist coordinates:

$$\vec{e}_{\hat{0}} = \frac{1}{\alpha}\left(\frac{\partial}{\partial t} + \omega\frac{\partial}{\partial\phi}\right) , \quad \vec{e}_{\hat{\phi}} = \frac{1}{\varpi}\frac{\partial}{\partial\phi} , \quad \vec{e}_{\hat{r}} = \frac{\sqrt{\Delta}}{\rho}\frac{\partial}{\partial r} , \quad \vec{e}_{\hat{\theta}} = \frac{1}{\rho}\frac{\partial}{\partial\theta} ; \tag{18a}$$

and their corresponding dual basis vectors (one-forms) are

$$\mathbf{e}^{\hat{0}} = \alpha\mathbf{d}t , \quad \mathbf{e}^{\hat{\phi}} = \varpi(\mathbf{d}\phi - \omega\mathbf{d}t) , \quad \mathbf{e}^{\hat{r}} = \frac{\rho}{\sqrt{\Delta}}\mathbf{d}r , \quad \mathbf{e}^{\hat{\theta}} = \rho\mathbf{d}\theta . \tag{18b}$$

a. Explain why the fact that the ZAMO 4-velocity, viewed as a 1-form, is $\mathbf{u} = -\mathbf{e}^{\hat{0}} = -\alpha\mathbf{d}t$ implies that the ZAMOs see the 3-surfaces of constant Boyer-Lindquist time t as their surfaces of physical "simultaneity", i.e. as their 3-dimensional space in a 3+1 split of space into space plus time.

b. When these ZAMOs perform a 3+1 split of the electromagnetic field tensor $\mathbf{F}$ into the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ that they measure, those $\mathbf{E}$ and $\mathbf{B}$ can be regarded as 4-vectors that lie in the ZAMOs' 3-space (surface of constant t). Show that, so regarded, the electric and magnetic fields are

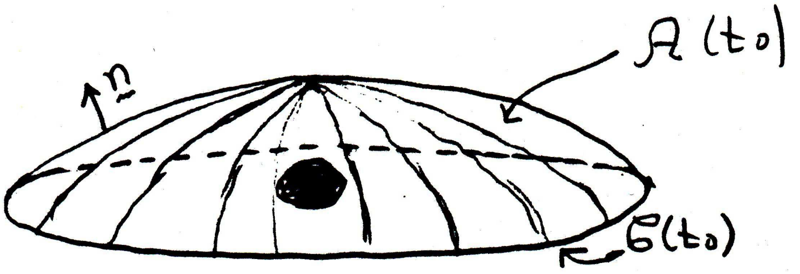
$$\vec{E} = \mathbf{F}(\_, \vec{u}) \ , \quad \vec{B} = - * \mathbf{F}(\_, \vec{u}) \ , \tag{19}$$

where the * denotes the dual, $*F_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}F^{\mu\nu}$. In component notation $E_\alpha = F_{\alpha\beta}u^\beta$ and $B_\alpha = -\frac{1}{2}\epsilon_{\alpha\beta\mu\nu}F^{\mu\nu}u^\beta$.

c. One can show that, in curved spacetime as in flat, Maxwell's equations imply that the integral of $*\mathbf{F}$ over a closed 2-dimensional surface in 4-dimensional spacetime vanishes. In other words, in index notation,

$$\int_{\partial\mathcal{S}_3} *F^{\alpha\beta} d^2\Sigma_{\alpha\beta} = 0 \ , \tag{20}$$

where $\partial\mathcal{S}_3$ is the closed 2-dimensional boundary of a 3-dimensional region $\mathcal{S}_3$. Evaluate this equation in terms of the electric and magnetic fields measured by the ZAMOs, for a closed 2-surface $\partial\mathcal{S}_3$ made up of the following pieces: (i) A circle $(r = r_c, \theta = \pi/2, 0 \le \phi \le 2\pi, t = t) \equiv \mathcal{C}(t)$ lying in the equatorial plane of the black hole, and moving forward in time by Δt: $t_o \le t \le t_o + \Delta t$. (ii) A 2-dimensional surface $\mathcal{A}(t_o)$ which lies in the 3-surface $t = t_o$, and stretches like a rubber sheet up over the black hole and is anchored on $\mathcal{C}(t_o)$, as shown in this drawing:



(iii)The same 2-surface $\mathcal{A}(t_o + \Delta t)$, but lying now in the 3-surface $t = t_o + \Delta t$. Show that Eq. (20) reduces, for Δt small, to a general relativistic version of Faraday's law of induction:

$$\frac{d}{dt}\int_{\mathcal{A}(t)} \mathbf{B}\cdot d\mathbf{A} = -\int_{\mathcal{C}(t)} \alpha\mathbf{E}\cdot d\mathbf{s} \ . \tag{21}$$

Here the notation is that of 3-dimensional vector analysis, with the vectors lying in the ZAMOs' 3-space (surface of constant t); $d\mathbf{A} = \mathbf{n}d\text{Area}$ is the vectorial area element on $\mathcal{A}(t)$ (with $\mathbf{n}$ the unit 3-vector outward-pointing normal to $\mathcal{A}(t)$; and $d\mathbf{s}$ is the 3-dimensional displacement vector along the curve $\mathcal{C}(t)$. Notice that this Faraday law is the same as in flat space, except for the lapse function α, which is needed because the derivative on the left side of the equation is with respect to coordinate time t and not ZAMO proper time.

d. Let $\mathbf{v}$ be the ordinary velocity (not 4-velocity) of the fluid as measured by a ZAMO. Show that the vanishing of the electric field in the rest frame of the fluid implies that

$$\mathbf{E} = -\mathbf{v}\times\mathbf{B} \ . \tag{22}$$

e. Use the results of parts c. and d. to show that $-\int_{\mathcal{C}(t)} \alpha \mathbf{E} \cdot d\mathbf{s}$ is equal to the rate per unit coordinate time t at which magnetic flux is carried inward, by the accretion disk's plasma, across the circle $\mathcal{C}(t)$.

This exercise illustrates how one derives the general relativistic versions of the equations of electrodynamics (expressed in terms of fields measured by the ZAMOs) that are used in analyzing black-hole magnetospheres, e.g. in Ref. 9.

8. ISCO in Schwarzchild and Equatorial ISCO in Kerr [12 Points]
This exercise discusses the inner-most stable circular orbits (ISCOs), which describes the inner edge of an accretion disk, and the final orbit of an "inspiral", which refers to the process in which a small compact object circulates around a much heavier object and gradually loses energy and angular momentum, and therefore spirals toward the heavier object, due to emission of gravitational waves (we will discuss this in Week 10).

a. Confirm that, for circular geodesics in Schwarzchild, we have

$$\frac{d\phi}{dt} = \sqrt{GM/r^3}$$

b. Argue that in general, for Schwarzchild and Kerr, for events that happen on circular geodesics once each period, the distance observer is going to see an angular frequency of

$$\omega = d\phi/dt \,,$$

c. For equatorial orbits in Kerr, using two conserved quantities $u_t = -\varepsilon$ and $u_\phi = \ell$, obtain the effective potential V_{eff}^2, and then obtain r_{ISCO} and ω_{ISCO}. Your result can be numerical, e.g., presented in a plot — which would be very easy to obtain using Mathematica. [Compare your plot with the one in Reading 2.]

d. In 2003, infrared flux fluctuations with a period of 17 minute is observed from the galactic center [R. Genzel et al., Nature 425, 934 (2003)]. If we assume 17 minute to be due to events that happen once each orbital period, we would then be able to say that ω_{ISCO} is larger than $2\pi/(17\text{min})$. Given a black-hole mass of $3.6 \times 10^6 M_\odot$ [obtained from Keplerian motion, e.g., F. Eisenhauer et al., Astrophys. J. 628 246-259 (2005)], what do we infer about a? In addition, if we take 17 minute as really arising from an equatorial ISCO orbit, what is the *actual* (or *proper*) frequency at which this fluctuation happens?

9. General Geodesics in Kerr Spacetime using the Hamilton-Jacobi Method.
Exercise 33.7 of MTW.