

WEEK 4: POST-NEWTONIAN RESUMMATION, SPINNING BINARIES, AND BLACK-HOLE PERTURBATION

(Dated: Jan 31, 2008)

I. READINGS AND REFERENCE MATERIALS

Here are some possibly useful materials for calculating Post-Newtonian waveforms from binary inspirals:

1. Zoology of different (non-spinning) Post-Newtonian prescriptions. [*Positive view*: span of different prescriptions characterizes the range in which we need to place our templates; *negative view*: difference between different prescriptions attest the failure/breakdown of the Post-Newtonian approach.]
 - (a) Damour, Iyer and Sathyaprakash, Phys. Rev. D **63** 044023 (2001), Phys. Rev. D **66** 027502 (2002) and Damour, Iyer, Jaranowski and Sathyaprakash, Phys. Rev. D **67** 064028 (2003).
 - (b) Buonanno, Chen and Vallisneri, Phys. Rev. D **67** 024016 (2003).
2. The Buonanno-Damour Effective One Body approach
 - (a) Initial formulation: Buonanno and Damour, Phys. Rev. D **59** 084006 (1999).
 - (b) 3PN: Damour, Jaranowski and Schaefer, Phys. Rev. D **62**, 084011 (2000).
 - (c) Application to transition from inspiral to plunge: Buonanno and Damour, Phys. Rev. D **62** 064015 (2000).
 - (d) Attempts to spinning binaries: Damour, Phys. Rev. D **64** 124013 (2001), and Buonanno, Chen and Damour, Phys. Rev. D **74** 104005 (2006).
3. Spin effect in equations of motion
 - (a) Laws of motion derived for objects with possibly strong gravity: Thorne and Hartle, Phys. Rev. D **31**, 1815 (1985).
 - (b) Post-Newtonian equations from point-particle results: Kidder, Will and Wiseman, Phys. Rev. D **47**, 4183 (1993).
4. Modulations in waveforms caused by spins
 - (a) Leading order waveforms and characteristic of precessions, as well as precession-induced modulations: Apostolatos, Cutler, Sussman and Thorne, Phys. Rev. D **49**, 6274 (1994).
 - (b) Short summary of how to calculate waveforms, and recap of the above result in shorter paragraphs: Secs. II and III of Buonanno, Chen and Vallisneri, Phys. Rev. D **67**104025 (2003).

Compared with Post-Newtonian waveform calculations, black hole perturbation theory can be learned from more accessible sources.

1. The original Regge-Wheeler paper, Phys. Rev. **108**, 1063, is very pleasant to read — despite some minor algebraic errors. [In less well-written papers, people wouldn't be able to find algebraic errors!]
2. General description of BH perturbation theory: Pages 178 – 195 of N.R. Lebovitz, W.H. Reid and P.O. Vandervoort, *Theoretical Principles in Astrophysics and Relativity*, 1978 (contribution by Thorne), see course website.
3. Thorne, Non-spherical gravitational collapse, a short review, in *Magic without Magic: John Archibald Wheeler*, where power-law tails are explained, and the *hoop conjecture* is formulated (see course website).
4. More mathematical treatment of tails, and its connection to BH being the unique final state of stellar collapse: R.H. Price, Phys. Rev. D **5**, 2419 (1972).
5. More complete and technical review on BH perturbation, V.P. Frolov and I.D. Novikov, *Black Hole Physics*, Chapter 4 (see course website).
6. Review article on black hole stability by prominent mathematicians: Felix Finster, Niky Kamran, Joel Smoller, Shing-Tung Yau, *Linear Waves in the Kerr Geometry: A Mathematical Voyage to Black Hole Physics*, arXiv:0801.1423.

II. PROBLEMS

This homework is due on Wednesday, February 6, before class. Again, the maximum possible grade is 50.

Problem 1. Effective One Body approach [25 Points] As you may have found out the initial EOB paper is very long and not very easy to read for non-experts. In fact, Damour, Jaranowski and Schaefer [reading 2(b)] gives a much easier account of how EOB can be constructed. In this exercise, let us try to recover the 2PN EOB dynamics.

- (a) As a preparation, show that given a metric $g_{\mu\nu}$, geodesic motion can be described by a non-relativistic Hamiltonian H obtained through

$$g^{\mu\nu}(x^\alpha)P_\mu P_\nu = -\mu^2, \quad P_0 = H + \mu, \quad P_j = \mathbf{P}, \quad x^0 = t, \quad x^j = \mathbf{Q} \quad (1)$$

- (b) Given effective metric coefficients A and D at 2PN order, obtain the corresponding non-relativistic Hamiltonian. Here the effective metric is given by

$$ds^2 = -A(R)dt^2 + \frac{D(R)}{A(R)}dR^2 + R^2(d\Theta^2 + \sin^2\Theta d\Phi^2) \quad (2)$$

- (c) Taylor-expand this non-relativistic Hamiltonian in \mathbf{P} and \mathbf{Q} . In fact, this Hamiltonian is really in terms of P_R^2 , P^2 and R , so the expansion is straightforward if you use Mathematica and assign $P_R^2 \rightarrow \varepsilon P_R^2$, $P^2 \rightarrow \varepsilon P^2$, and $1/R \rightarrow \varepsilon/R$, and expand in ε .
- (d) Introduce additional parameters in the canonical transformation of Damour, Jaranowski and Schaefer, and obtain the canonically-transformed-Taylor-expanded non-relativistic Hamiltonian.
- (e) Introduce final parameters in the energy mapping relation, and finally match all “effective parameters” to PN parameters, and obtain the 2PN EOB, in particular the A and D functions. (They are functions of R and mass ratio η .)
- (f) Study the effective potential of a particle moving in this deformed space time. Do we have an ISCO?

Problem 2. Gravitational Waveforms from a spinning binary [30 Points] Let us try to get the simplest modulated gravitational waveform from spinning binaries. We will use equations summarized in reading 4(b).

- (a) For evolution of orbital frequency, take the leading order of Eq. (1). Let us forget about the ending of the inspiral due to various effects, and instead just evolve ω to infinity. The *accumulated orbital phase* is determined by Eq. (10).
- (b) For evolution of spin directions, take Eqs. (2) and (3), but let us only include one spin, presumably on the object with higher mass.
- (c) For evolution of the direction of the orbital plane, take Eq. (9), where $\hat{\mathbf{L}}_N$ is the normal direction of the orbital plane. Also, only one spin.
- (d) Up to this stage, you should be able to investigate effects like how the orbital plane ($\hat{\mathbf{L}}_N$) precesses, and how the spin precesses. Try to evolve several interesting configurations, verify the following:
- The total angular momentum stays roughly constant in orientation, $\hat{\mathbf{J}}$
 - Then project $\hat{\mathbf{L}}_N$ to the plane orthogonal to the constant $\hat{\mathbf{J}}$, and see how precession phase depends on orbital frequency.
 - In order to comparison, let us consider $10 + 1.4$ binaries, with initial frequency of 10 Hz, and initial angle between \mathbf{S} and $\hat{\mathbf{L}}_N$ equal to 45, 90, and 135 degrees. Interesting information to obtain would be: what is the number of precession cycles till the end of the inspiral?
- (e) In order to get the waveform, we also need to know the instantaneous relative separation vector between the two objects. For this see Sec. IIIC of the paper.
- (f) For the above same sample configurations, place the detector either in the direction of the initial BH spin direction, at an orthogonal direction, and somewhere in between, and plot the two polarizations.

Problem 3. Regge-Wheeler Gauge and Regge-Wheeler Equation [20 Points] Read the Regge-Wheeler paper, and try to understand it in the following steps:

- (a) Explain why Eqs. (12) and (13) give the most general metric perturbation.
- (b) Argue in detail why metric perturbations Eqs. (12) and (13) can be put into forms of Eqs. (18) and (20). Pay attention to the difference between covariant derivative when we confine to (θ, ϕ) , and the covariant derivative when we consider all (t, r, θ, ϕ) .
- (c) For odd wave, obtain Eqs. (22), and subsequently the Regge-Wheeler equation for Q . [Perhaps it would be helpful to use tensor-manipulation softwares.]

Problem 4. Black-Hole Stability, and Quasinormal mode with highest quality factor. [15 Points] For the Regge-Wheeler wave equation

$$\frac{\partial^2 Q}{\partial t^2} - \frac{\partial^2 Q}{\partial r_*^2} + VQ = 0 \quad (3)$$

- (a) Define energy density as

$$\epsilon \equiv \left(\frac{\partial Q}{\partial t} \right)^2 + \left(\frac{\partial Q}{\partial r_*} \right)^2 + VQ^2 \quad (4)$$

and prove the energy conservation relation

$$\frac{d}{dt} \int_a^b \epsilon dr_* = \frac{\partial Q}{\partial r_*} \frac{\partial Q}{\partial t} \Big|_a^b \quad (5)$$

Explain this equation physically.

- (b) Write down the boundary condition for quasinormal modes. Make the transformation of variable $u(r_*) \equiv Q(ir_*)$, and

$$-\frac{\partial^2 Q}{\partial r_*^2} + [-\omega^2 + V]Q = 0, \quad Q \sim e^{\pm i\omega r_*}, \quad (r_* \rightarrow \pm\infty), \quad \text{Re}(\omega) > 0, \quad (6)$$

becomes

$$-\frac{\partial^2 u}{\partial r_*^2} + [\omega^2 - V]u = 0, \quad u \sim e^{\mp\omega r_*}, \quad (r_* \rightarrow \pm\infty), \quad \text{Re}(\omega) > 0, \quad (7)$$

or

$$-\frac{\partial^2 u}{\partial r_*^2} + \tilde{V}u = Eu, \quad u \sim e^{\mp\omega r_*}, \quad (r_* \rightarrow \pm\infty), \quad \text{Re}(\omega) > 0, \quad (8)$$

with the identification $\tilde{V} \equiv -V$, and $E \equiv -\omega^2$. Then explain how this maps the BH quasi-normal mode problem into solving modes of particle in a finite potential well. Here the difference is that we allow complex eigenvalues E .

- (c) For large l , explain why it is reasonable to expect the highest-quality-factor mode to have $M\omega \sim l/\sqrt{27}$, and why that mode would be almost lossless.
- (d) For highly damped quasinormal modes, people observe (independently from l)

$$8\pi M\omega_n = \ln 3 + 2\pi i(n + 1/2). \quad (9)$$

[Some people, e.g., Hod, PRL 81, 4294, (1998), and Maggiore, arxiv:0711.3145, believe that these are high-order excitations of the black-hole horizon, and are therefore intimately connected to the black-hole entropy.] Let us try to recover this sequence of modes in the Schrödinger problem with potential well. Unfortunately, in our real problem, the potential well is infinite, and some delicate mathematical treatment seems to be needed. For a one-dimensional square potential well with (constant) depth V and size L , show that there exist energy levels like

$$E \sim \left(\frac{2n\pi}{L} \right)^2 - V, \quad (10)$$

then show that this agrees qualitatively with Eq. (9).