MOMENTS OF AN EXPONENTIAL SUM RELATED TO THE DIVISOR FUNCTION

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ABSTRACT. We use the circle method to obtain tight bounds on the L^p norm of an exponential sum involving the divisor function for $p > 2$.

1. INTRODUCTION

Let $X \geq 1$ be sufficiently large. For a function $f : \mathbb{N} \to \mathbb{C}$, let

$$M_f(\alpha) = \sum_{n \leq X} f(n) e(n\alpha)$$

where as usual, $e(\alpha) := e^{2\pi i \alpha}$. Information on the structure of $f(n)$ can be obtained by studying the size of L^p-integrals of $M_f(\alpha)$, and bounds on them are often useful in applications of the circle method. Write

$$I_f(p) := \int_0^1 |M_f(\alpha)|^p d\alpha. \tag{1.1}$$

In the case $p = 1, f = \tau$, it was shown in [GP] that

$$\sqrt{X} \ll I_\tau(1) \ll \sqrt{X} \log X. \tag{1.2}$$

where

$$\tau(n) := \sum_{d|n} 1.$$

For sequences other than $\tau(n)$, similar results have been established in the case $p = 1$. For example, with μ the Möbius function, we have that $X^{1/6} \ll I_\mu(1) \ll X^{1/2}$ where the upper bound follows from Parseval's identity, and the lower bound follows from Theorem 3 in [BR]. Estimates for $I_f(1)$ in the case f is an indicator function for the primes have been obtained by Vaughan [Va1] and Goldston [Go], and in the case f is the indicator function for integers not divisible by the rth power of any prime by Balog and Ruzsa [BR] (in fact, a result of Keil [Ke] finds the exact order of magnitude for all moments but $1 + \frac{1}{r}$ in which case the exact order of magnitude is found within a factor of $\log X$).
In this paper, we shall focus on the case $f = \tau$, the divisor function. Note that we have that by Parseval's identity

$$I_\tau(2) = \sum_{n \leq X} \tau(n)^2 \sim \frac{1}{\pi^2} X (\log X)^3. \tag{1.3}$$

In this paper, we shall obtain tight estimates on $I_\tau(p)$ for $p > 2$. In particular, we prove the following result.

Theorem 1.1. *We have that for $p > 2$*

$$\int_0^1 |M_\tau(\alpha)|^p d\alpha \asymp_p X^{p-1}(\log X)^p. \tag{1.4}$$

Throughout this paper, all implied constants will be assumed to depend only on p unless otherwise specified.

2. Preliminaries and setup

Note that we have that

$$M_\tau(\alpha) = \sum_{n\le X} \tau(n)e(n\alpha) = \sum_{uv\le X} e(\alpha uv) = 2\sum_{\substack{uv\le X\\ u<v}} e(\alpha uv) + \sum_{\substack{uv\le X\\ u=v}} e(\alpha uv) = 2T(\alpha)+E(\alpha) \tag{2.1}$$

where

$$T(\alpha) := \sum_{u\le X^{1/2}} \sum_{u<v\le X/u} e(\alpha uv), E(\alpha) := \sum_{u\le X^{1/2}} e(\alpha u^2).$$

Also, let

$$v(\beta) := \sum_{n\le X} e(n\beta).$$

We record the following well-known bound on $v(\beta)$ which we will use later.

Lemma 2.1. *We have that for $\beta \notin \mathbb{Z}$, $v(\beta) \asymp \min(X, \|\beta\|^{-1})$ where for $\alpha \in \mathbb{R}$, we let $\|\alpha\| := \inf_{n\in\mathbb{Z}} |\alpha - n|$.*

In addition, we shall also use the following result on moments of $v(\beta)$.

Lemma 2.2. *For $p > 2$, we have that*

$$\int_0^1 |v(\beta)|^p \asymp X^{p-1}.$$

Proof. Note that by Lemma 2.1, we have that

$$\int_0^1 |v(\beta)|^p d\beta \ge \int_{-X^{-1}}^{X^{-1}} |v(\beta)|^p d\beta \gg \int_{X^{-1}}^{X^{-1}} X^p d\beta \gg X^{p-1}. \tag{2.2}$$

In addition, note that for positive integers s, by considering the underlying Diophantine system, we have that

$$\int_0^1 |v(\beta)|^{2s} d\beta \sim C_s X^{2s-1}$$

so the desired result follows from Hölder's inequality. □

We will use the circle method to prove the main result. To that end, let

$$\mathfrak{M}(q,a) = \{\alpha \in [0,1] : |q\alpha - a| \le PX^{-1}\}$$

with $P = X^\nu$ for $\nu > 0$ sufficiently small, and let

$$\mathfrak{M} = \bigcup_{q\le P} \bigcup_{\substack{a=1\\ (a,q)=1}}^{q} \mathfrak{M}(q,a), \mathfrak{m} = [0,1] \setminus \mathfrak{M}.$$

For any measurable $\mathfrak{B} \subseteq [0,1)$, let

$$I_f(p;\mathfrak{B}) := \int_{\mathfrak{B}} |M_f(\alpha)| d\alpha.$$

We shall prove Theorem 1.1 by using the fact that $I_\tau(p) = I_\tau(p;\mathfrak{M}) + I_\tau(p;\mathfrak{m})$, showing that $I_\tau(p;\mathfrak{m}) = o(X^{p-1}(\log X)^p)$ and showing that $I_\tau(p;\mathfrak{M}) \asymp X^{p-1}(\log X)^p$.

3. The minor arcs

Our bound on the minor arcs will depend on the following result, which is nontrivial for $X^\varepsilon \ll q \ll X^{1-\varepsilon}$.

Proposition 3.1. *If $|q\alpha - a| \leq q^{-1}$ for some $(a,q) = 1$, $q \geq 1$, then*

$$M_\tau(\alpha) \ll X \log(2Xq)(q^{-1} + X^{-1/2} + qX^{-1}). \tag{3.1}$$

Proof. We have that by (2.1) and the trivial bound $|E(\alpha)| \leq X^{1/2}$

$$M_\tau(\alpha) = 2T(\alpha) + O(X^{1/2})$$

so it suffices to show that $T(\alpha) \ll X \log(2Xq)(q^{-1} + X^{-1/2} + qX^{-1})$, since we can absorb the $O(X^{1/2})$ into the bound since $X \log(2Xq)(q^{-1} + X^{-1/2} + qX^{-1}) \gg X^{1/2} \log X$.

To this end, note that by the triangle inequality

$$|T(\alpha)| \leq \sum_{u \leq X^{-1}} \left| \sum_{u < v \leq X/u} e(\alpha uv) \right| \ll \sum_{u \leq X^{1/2}} \min(X/u, \|\alpha u\|^{-1}).$$

The desired result then follows from Lemma 2.2 in [Va]. □

From this, the following result follows.

Lemma 3.1. *We have that*

$$I_\tau(p;\mathfrak{m}) \ll X^{p-1-\nu/2}(\log X)^4. \tag{3.2}$$

Proof. Note that we have that

$$\int_{\mathfrak{m}} |M_\tau(\alpha)|^p d\alpha \leq \left(\sup_{\alpha \in \mathfrak{m}} |M_\tau(\alpha)|\right)^{p-2} \int_{\mathfrak{m}} |M_\tau(\alpha)|^2 d\alpha \ll X(\log X)^3 \left(\sup_{\alpha \in \mathfrak{m}} |M_\tau(\alpha)|\right)^{p-2}.$$

Suppose that $\alpha \in \mathfrak{m}$. Then, by Dirichlet's theorem, we have that there exist a, q s.t. $(a,q) = 1, q \leq P^{-1}X, |q\alpha - a| \leq P^{-1}X$, so it follows that $q > P$. Then, by Proposition 3.1, we have that $|M_\tau(\alpha)| \ll X^{1-\nu/2} \log X$, and the desired result follows. □

Now, we proceed to estimate the major arcs. To that end, we first record the following estimate.

Proposition 3.2. *For $(a,q) = 1$, $q \geq 1$, we have*

$$\sum_{n \leq X} \tau(n) e\left(\frac{an}{q}\right) = \frac{X}{q}\left(\log \frac{X}{q^2} + 2\gamma - 1\right) + O((X^{1/2} + q) \log 2q).$$

Proof. This is shown in the proof of Lemma 2.5 in [PV]. We shall reproduce its proof below. Note that we have that

$$\sum_{n\le X}\tau(n)e\left(\frac{an}{q}\right)=\sum_{u\le X^{1/2}}\left(\sum_{v\le X/u}2-\sum_{v\le X^{1/2}}1\right)e(auv/q).$$

For $q\nmid u$, we have that the inner sums are $\ll \|au/q\|^{-1}$. The contribution from the remaining terms is then

$$\frac{X}{q}\left(\log\frac{X}{q^2}+2\gamma-1\right)+O(X^{1/2})$$

from which the desired result follows. □

Now, it follows then from this and partial summation that for $\alpha\in\mathfrak{M}(q,a)$, we have

$$M_\tau(\alpha)=\frac{1}{q}\left(\log\frac{X}{q^2}+2\gamma-1\right)v(\alpha-a/q)+O(X^{1/2+\nu}\log X). \tag{3.3}$$

Therefore, we have that (by using the binomial theorem for $p\in\mathbb{Z}^+$, and then using Hölder's inequality to bound the remaining error terms)

$$|M_\tau(\alpha)|^p=q^{-p}(\log X-2\log q+2\gamma-1)^p|v(\alpha-a/q)|^p+O(X^{p-1/2+\nu}(\log X)^p)$$

so it follows that

$$\int_{\mathfrak{M}}|M_\tau(\alpha)|^p d\alpha= \tag{3.4}$$

$$\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{-PX^{-1}}^{PX^{-1}}q^{-p}(\log X-2\log q+2\gamma-1)^p|v(\alpha-a/q)|^p d\beta+O(X^{p-3/2+4\nu}(\log X)^p)$$

$$=\mathfrak{S}(X,P)\int_{-PX^{-1}}^{PX^{-1}}|v(\beta)|^p d\beta+O(X^{p-3/2+4\nu}(\log X)^p)$$

where

$$\mathfrak{S}(X,P):=\sum_{q\le P}\varphi(q)q^{-p}(\log X-2\log q+2\gamma-1)^p.$$

It is easy to show that

$$\mathfrak{S}(X,P)\asymp(\log X)^p. \tag{3.5}$$

Also, note that since $|v(\beta)|\le\min(X,\|\beta\|^{-1})$, we have that

$$\int_{-PX^{-1}}^{PX^{-1}}|v(\beta)|^p d\beta\gg\int_0^{1/(4X)}X^p d\beta\gg X^{p-1}.$$

By considering the underlying diohantine equation, it is quite easy to show that for positive integers $s>0$, we have that

$$\int_0^1|v(\alpha)|^{2s}d\alpha\sim C_sX^{2s-1}$$

for some $C_s>0$. It therefore follows that by Hölder's inequality since $p>2$

$$\int_0^1|v(\beta)|^p d\beta\asymp X^{p-1}.$$

Theorem 1.1 then follows.

References

[BR] A. Balog, I. Z. Ruzsa, *On the exponential sum over r-free integers.* Acta Math. Hungar. 90 (2001), no. 3, 219230.

[GP] D. A. Goldston, M. Pandey, *On the L^1 norm of an exponential sum involving the divisor function*, preprint.

[Go] D. A. Goldston, *The major arcs approximation of an exponential sum over primes*. Acta Arith. 92 (2), 169 (2000).

[Ke] E. Keil, *Moment estimates for exponential sums over k-free numbers*. Int. J. Number Theory, 9, (2013), 607619.

[PV] P. Pongsriiam and R. C. Vaughan, *The divisor function on residue classes I*, Acta. Arith. 168 (2015), 369381

[Va1] R. C. Vaughan, *The L^1 mean of exponential sums over primes*. Bull. London Math. Soc. 20, 121 123 (1988).

[Va] Vaughan, R. C., *The Hardy-Littlewood Method* Cambridge Tracts in Mathematics. 125 (2nd ed.). Cambridge University Press, 1997.