A. S. Kechris: Global Aspects of Ergodic Group Actions; Corrections and Updates (December 18, 2025)

Page x, line 3: Replace “the equivalence” by “an ergodic equivalence".

Page 8, last 7 lines of the proof of 2.5: Replace ξ_i by η_i.

Page 12, line 3-: Add “ .” at the end.

Page 19, Theorem 3.13: This was originally proved in H.A. Dye, On groups of measure preserving transformations, I, *Amer. J. Math.,* **81(1)** (1959), 119–159.

Page 20, Theorem 4.1: This was originally proved in H.A. Dye, On groups of measure preserving transformations, II, *Amer. J. Math.,* **85(4)** (1963), 551–576.

Page 22, line 20-: François Le Maître pointed out that in [BG1] the authors prove this result for the class of measure-class preserving equivalence relations but the older result for the case of measure preserving equivalence relations is due to H.A. Dye, On groups of measure preserving transformations, II, *Amer. J. Math.,* **85** (1963), 551–576, Proposition 5.1.

Page 26: An update concerning the results in the paragraph following 4.12:

Matui has shown that for ergodic hyperfinite E, $t([E]) = 2$, and $t([E_n]) \leq 2(n+1)$. Independently Marks also proved that $t([E]) = 2$ and moreover showed that $t([E_n]) \leq 2n$ and if E_n is induced by a free modular action of F_n, then $t([E_n]) = n+1$.

Finally Le Maître proved that for *every ergodic* measure preserving E one has the formula $t([E]) = \lfloor C_\mu(E) \rfloor + 1$, where $\lfloor C_\mu(E) \rfloor$ is the integer part of the cost of E, and therefore $t([E_n]) = n+1$. He also obtained a related formula when E is aperiodic but not necessarily ergodic.

Page 30, line 1-: After "3.14" add "and 2.6".

Page 47, line 7: "$\ker(\pi_1) = \ker(\pi_2)$" should be "$\ker(\bar{\pi}_1) = \ker(\bar{\pi}_2)$".

Page 67, line 18: Delete "ergodic".

Page 71, 10.9: Add in the statement of the theorem that Γ is an infinite countable group.

Page 71, line 4: Before "Let" add "By 10.8, we can assume that a is free.".

Page 71, last sentence of the first paragraph of the proof of 10.9: Replace this sentence by:

Proposition. *The measure μ is in the closure of the set of probability measures of the form $\sum_{i=1}^m \alpha_i \mu_i$, with $0 \leq \alpha_i \leq 1, \sum_{i=1}^m \alpha_i = 1$, and each μ_i a non-atomic, a-invariant, ergodic probability measure.*

We give below a proof of this proposition.

Proof. We start with the following lemma:

Lemma. *Let Y be a standard Borel space and ρ a probability measure on Y. Let $f_i \colon Y \to \mathbb{R}$, $1 \leq i \leq n$, be bounded Borel functions, and let $\epsilon > 0$. Then there is a finite support probability measure ν on Y such that*

$$\Big| \int f_i \mathrm{d}\rho - \int f_i \mathrm{d}\nu \Big| < \epsilon, 1 \leq i \leq n.$$

Proof. We can assume that Y is a Polish space. Let M be the maximum of the l^∞-norms of the functions $f_i, 1 \leq i \leq n$. Let also K be compact such that each $f_i|K, 1 \leq i \leq n$, is continuous, and $\rho(K)$ is bigger than $1 - \frac{\epsilon}{4M}$. Recall that the finite support probability measures are dense in the weak*-topology of the space $P(K)$ of probability measures on K. Let $\rho_K = \frac{\rho|K}{\rho(K)}$. Then the set

$$\{\nu \in P(K) \colon \Big| \int (f_i|K) \mathrm{d}\rho_K - \int (f_i|K) \mathrm{d}\nu \Big| < \frac{\epsilon}{2}, 1 \leq i \leq n,$$

is open in $P(K)$, so there is a finite support $\nu \in P(K)$ such that

$$\Big| \int (f_i|K) \mathrm{d}\rho_K - \int (f_i|K) \mathrm{d}\nu \Big| < \frac{\epsilon}{2}, 1 \leq i \leq n.$$

Clearly

$$\int (f_i|K) \mathrm{d}\nu = \int f_i \mathrm{d}\nu$$

and

$$\int (f_i|K)\mathrm{d}\rho_K = \frac{1}{\rho(K)}\int_K f_i\mathrm{d}\rho,$$

so

$$|\frac{1}{\rho(K)}\int_K f_i\mathrm{d}\rho - \int f_i\mathrm{d}\nu| < \frac{\epsilon}{2}.$$

But

$$\begin{aligned} &|\frac{1}{\rho(K)}\int_K f_i\mathrm{d}\rho - \int f_i\mathrm{d}\rho| \\ = &|\frac{1}{\rho(K)}\int_K f_i\mathrm{d}\rho - \int_K f_i\mathrm{d}\rho - \int_{Y\setminus K} f_i\mathrm{d}\rho| \\ \leq &(\frac{1}{\rho(K)} - 1)\rho(K)\cdot M + \rho(Y\setminus K)\cdot M \\ < &\frac{\epsilon}{4} + \frac{\epsilon}{4}, \end{aligned}$$

so

$$|\int f_i\mathrm{d}\rho - \int f_i\mathrm{d}\nu| < \epsilon, 1 \leq i \leq n.$$

□

We use this lemma to complete the proof of the proposition.

Denote by E the equivalence relation induced by a and let

$$\pi\colon X \to \mathcal{EI}_E$$

be the ergodic decomposition of E, see A.S. Kechris and B.D. Miller, Topics in Orbit Equivalence, Springer, Lectures Notes in Mathematics, **1852**, Theorem 3.3. Let $\rho = \pi_*\mu$.

Lemma. *Let $\mathcal{A} \subseteq \mathcal{EI}_E$ be the set of atomic measures. Then $\rho(\mathcal{A}) = 0$.*

Proof. Let $Z \subseteq X$ be an a-invariant Borel set with $\mu(Z) = 1$ on which a acts freeely. If $X_e = \{x\colon \pi(x) = e\}$, then for each atomic $e \in \mathcal{EI}_E$, $X_e \cap Z = \emptyset$, so $\rho(\mathcal{A}) = \mu(\bigcup_{e\in\mathcal{A}} X_e) = 0$. □

Let $f_i\colon X \to \mathbb{R}$, $1 \leq i \leq n$, be bounded Borel functions, and let $\epsilon > 0$. We claim that there are non-atomic $\mu_1, \dots, \mu_m \in \mathcal{EI}_E$ and $\alpha_1, \dots, \alpha_m$ with $0 \leq \alpha_i \leq 1$ and $\sum_{i=1}^m \alpha_i = 1$, such that if $\nu = \sum_{i=1}^m \alpha_i\mu_i$, then

$$|\int f_i\mathrm{d}\mu - \int f_i\mathrm{d}\nu| < \epsilon, 1 \leq i \leq m.$$

which completes the proof of the proposition.

To see this let $g_i \colon \mathcal{EI}_E \to \mathbb{R}$ be defined by $g_i(e) = \int f_i \mathrm{d}e$. These are bounded Borel functions, so we can apply the first lemma above to find $\mu_1, \ldots, \mu_m \in \mathcal{EI}_E$ and $\alpha_1, \ldots, \alpha_m$ with $0 \leq \alpha_i \leq 1$ and $\sum_{i=1}^m \alpha_i = 1$, such that if $\sigma = \sum_{i=1}^m \alpha_i \delta_{\mu_i}$, then

$$\left| \int g_i \mathrm{d}\rho - \int g_i \mathrm{d}\sigma \right| < \epsilon, 1 \leq i \leq n.$$

Note that by the proof of that lemma, we can take the $\mu_1, \ldots, \mu_m \in \mathcal{EI}_E$ to be non-atomic by choosing K in that proof to avoid the atomic $e \in \mathcal{EI}_E$.

Now

$$\int g_i \mathrm{d}\rho = \int g_i(e) \mathrm{d}\rho(e) = \int (f_i \mathrm{d}e) \mathrm{d}\rho(e) = \int f_i \mathrm{d}\mu.$$

and

$$\int g_i \mathrm{d}\sigma = \sum_{i=1}^m \alpha_i g_i(\mu_i) = \sum_{i=1}^m \alpha_i \int f_i \mathrm{d}\mu_i = \int f_i \mathrm{d}\nu,$$

where $\nu = \sum_{i=1}^m \alpha_i \mu_i$, so we are done. □

Page 73, line 7-: Replace “iff" by “ $\implies$ ".

Page 73, line 1-: Replace “$\Gamma \leq Z(\Delta)$" by “$\Gamma \leq C_\Delta(T)$ = the centralizer of T in Δ".

Page 75, Remark: Robin Tucker-Drob answered the question at the end of this remark by showing that $C^*(\Gamma) = C^{**}(\Gamma)$ for all infinite groups Γ. The proof is as follows:

It is enough to show that if Γ has property (T), then $C(\Gamma) = C^*(\Gamma) = C^{**}(\Gamma)$. Indeed, suppose that Γ has property (T). Then Γ is finitely generated (see [BdlHV, 1.3.1]), so by Corollary 10.14 we have $C(\Gamma) = C^*(\Gamma)$. It remains to show that $C(\Gamma) = C^{**}(\Gamma)$. By Theorem 13.1 the set A, consisting of all actions $a \in \mathrm{FRERG}(\Gamma, X, \mu)$ with weakly dense conjugacy class in $\mathrm{FRERG}(\Gamma, X, \mu)$, is dense G_δ in $(\mathrm{FRERG}(\Gamma, X, \mu), w)$. Fix $a \in A$. Let $b \in \mathrm{FR}(\Gamma, X, \mu)$ and let $\mu = \int_Z \mu_z \, d\eta(z)$ be the ergodic decomposition of b, so η-almost every measure μ_z is b-invariant and ergodic. Let b_z denote the action of Γ on (X, μ_z), so that b_z is almost surely ergodic. Since b is free, η-almost every b_z is free. Hence, for η-almost every $z \in Z$ we have $b_z \prec a$. This implies $b \prec i_Z \times a$, where i_Z is the identity action of Γ on (Z, η). Since Γ is finitely generated Corollary 10.14 gives $C(i_Z \times a) \leq C(b)$;

since $C(i_Z \times a) = C(a)$, we have have $C(a) \leq C(b)$. As $b \in \mathrm{FR}(\Gamma, X, \mu)$ was arbitrary, this shows that $C(a) = C(\Gamma)$. Since $a \in A$ was arbitrary and A is dense G_δ in $(\mathrm{FRERG}(\Gamma, X, \mu), w)$, this shows that $C^{**}(\Gamma) = C(\Gamma)$.

Page 79, lines 4-5: R. Tucker-Drob, Shift-minimal groups, fixed price 1, and the unique trace property, *arXiv: 1211.6395v3*, Corollary 6.22, shows that for infinite Γ and $a \prec b$, a free:

$$C(b) < \infty \implies C(b) \leq C(a),$$
$$E_b \text{ treeable} \implies C(b) \leq C(a).$$

Page 87, line 18: Add “)" at the end of the formula.

Page 90, 12.11: Replace [HJ4] by [HJ5].

Page 90, Proof of 13.1: Here view (by picking representatives) the a_n, a as Borel actions and not equivalence classes of Borel actions up to equality a.e.

Page 91, last Remark: Add:

It is shown in D. Kerr and H. Li, *Ergodic Theory*, Springer, 2016, Theorem 4.45, using the results of D.S. Ornstein and B. Weiss, Entropy and isomorphism theorems for actions of amenable groups, *J. Analyse Math.* **48** (1987), 1–141, that if Γ is amenable, then every action in $\mathrm{FR}(\Gamma, X, \mu)$ has dense conjugacy class in $(\mathrm{FR}(\Gamma, X, \mu), u)$.

Page 114, line 9-: Replace “lemma" by “theorem".

Page 123, line 7-: add “sofic" before Γ.

Page 123, (IIIb) and (IVb): G. Hjorth and A. Törnquist, The conjugacy relation on unitary representations, *Math. Res. Lett.*, **19(3)** (2012), 525–535, have shown that unitary equivalence for any Γ is Borel, in fact $\mathbf{\Pi^0_3}$.

In E. Gardella and M. Lupini, The complexity of conjugacy, orbit equivalence, and von Neumann equivalence of actions of nonamenable groups, *arXiv:1708.01327v2*, it is shown that isomorphism and orbit equivalence on free, ergodic (in fact even weak mixing) actions are not Borel for all nonamenable groups.

Page 131, 3 lines before (B): generically $\to$ generally.

Page 141, line 4 of the proof of 21.4: add period.

Page 144, line 6-: The third "n" from the left should be "N".

Page 184, last lines of 30.4, 30.5: G-superrigid $\rightarrow$ G-cocycle superrigid.

Page 185, third line of 30.6: G-superrigid $\rightarrow$ G-cocycle superrigid.

Page 186, line 10-: F_2-superrigid $\rightarrow$ F_2-cocycle superrigid.

Page 186, line 2-: G-superrigid $\rightarrow$ G-cocycle superrigid.

Page 208, line 7 of (B): positive definite $\rightarrow$ positive-definite.

Page 214, line 14: After "claim" add "applied to $e_1, \ldots, e_p$ and a given open set $W' \subseteq W$".

Page 234: add to index
i_Γ, 67